November $\cancel{8}$ 10 Math 2306 sec. 51 Spring 2023

Section 15: Shift Theorems

Suppose we wish to evaluate $\mathscr{L}^{-1}\left\{\frac{2}{(s-1)^3}\right\}$. Does it help to know that $\mathscr{L}\left\{t^2\right\}=\frac{2}{s^3}$?

Note that by definition

$$e^{-st}\cdot e^{t} = e^{-st+t} = e^{-(s-1)t}$$

$$\mathscr{L}\left\{e^t t^2\right\} = \int_0^\infty e^{-st}e^t t^2\,dt$$

$$= \int_0^\infty e^{-(s-1)t}\, t^2\,dt$$

If we let $w=s-1$

$$= \int_0^\infty e^{-wt}\, t^2\,dt$$

$$= \frac{2!}{w^3} = \frac{2!}{(s-1)^3}$$

Shift (or translation) in s.

Theorem: Suppose $\mathscr{L}\{f(t)\} = F(s)$. Then for any real number a

$$\mathscr{L}\left\{e^{at}f(t)\right\} = F(s-a).$$

We can state this in terms of the inverse transform. If $F(s)$ has an inverse Laplace transform, then

$$\mathscr{L}^{-1}\{F(s-a)\} = e^{at}\mathscr{L}^{-1}\{F(s)\}.$$

We call this a **translation** (or a **shift**) in s theorem.

Example:

Suppose $f(t)$ is a function whose Laplace transform[1]

$$F(s) = \mathscr{L}\left\{f(t)\right\} = \frac{1}{\sqrt{s^2+9}}$$

Evaluate

$$\mathscr{L}\left\{e^{-2t}f(t)\right\} = \frac{1}{\sqrt{(s+2)^2+9}}$$

$a = -2$

$F(s-(-2)) = F(s+2)$

[1] It's not in our table, but this is an actual function known as a *Bessel function*.

Examples: Evaluate

(a) $\mathscr{L}\{t^6 e^{3t}\} = \dfrac{6!}{(s-3)^7}$

Let $F(s) = \mathscr{L}\{t^6\} = \dfrac{6!}{s^7}$

$a = 3$, we need $F(s-3)$

Examples: Evaluate

(b) $\mathscr{L}\{e^{-t}\cos(t)\}$ $= \dfrac{s+1}{(s+1)^2+1}$

$F(s) = \mathscr{L}\{\cos t\} = \dfrac{s}{s^2+1^2}$

$a = -1$ $\quad F(s-(-1)) = F(s+1)$

(c) $\mathscr{L}\{e^{-t}\sin(t)\}$ $= \dfrac{1}{(s+1)^2+1}$

$F(s) = \mathscr{L}\{\sin t\} = \dfrac{1}{s^2+1^2}$

Inverse Laplace Transforms (completing the square)

(a) $\mathscr{L}^{-1}\left\{\dfrac{s}{s^2+2s+2}\right\}$

Since s^2+2s+2 is irreducible, partial fractions is not an option. We'll complete the square.

$$s^2+2s+2 = s^2+2s+1-1+2 = (s+1)^2+1$$

$$\frac{s}{s^2+2s+2} = \frac{s}{(s+1)^2+1}$$ we need $s+1$ everywhere there is an s.

Use $s = s+1-1$

$$= \frac{s+1-1}{(s+1)^2+1} = \frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+2s+2}\right\} = \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}$$

$$= e^{-1t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - e^{-1t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= e^{-t}\cos t - e^{-t}\sin t$$

Inverse Laplace Transforms (repeated linear factors)

(b) $\mathscr{L}^{-1}\left\{\dfrac{1+3s-s^2}{s(s-1)^2}\right\}$

Partial fractions

$s(s-1)^2$

$$\frac{-s^2+3s+1}{s(s-1)^2} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{(s-1)^2}$$ clear fractions

$$-s^2+3s+1 = A(s-1)^2 + Bs(s-1) + Cs$$

$$= A(s^2-2s+1) + B(s^2-s) + Cs$$

$$-1s^2+3s+1 = (A+B)s^2 + (-2A-B+C)s + A$$

$A+B=-1 \quad \Rightarrow \quad B=-1-A=-2$

$-2A-B+C=3 \qquad C=3+B+2A=3$

$A=1$

$$\frac{-s^2+3s-1}{s(s-1)^2} = \frac{1}{s} - \frac{2}{s-1} + \frac{3}{(s-1)^2}$$

$$\mathcal{L}^{-1}\left\{\frac{1+3s-s^2}{s(s-1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1!}{(s-1)^2}\right\}$$

$$= 1 - 2e^{1t} + 3e^{1t}\mathcal{L}^{-1}\left\{\frac{1!}{s^2}\right\}$$

$$= 1 - 2e^{t} + 3te^{t}$$