

## Section 16: Laplace Transforms of Derivatives and IVPs

Use the Laplace transform to solve the system of equations

$$\begin{aligned}x''(t) &= y, & x(0) &= 1, & x'(0) &= 0 \\ y'(t) &= x, & y(0) &= 1\end{aligned}$$

We took the transform and used Crammer's Rule to get to the solution

$$\begin{aligned}X(s) &= \frac{2/3}{s-1} + \frac{1/3(s-1)}{s^2+s+1} \\ Y(s) &= \frac{2/3}{s-1} + \frac{1/3(s+2)}{s^2+s+1}\end{aligned}$$

$$\frac{k}{s^2+k^2}$$
$$\frac{s}{s^2+k^2}$$

The irreducible quadratic denominator

$$s^2 + s + 1 = \left(s + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2.$$

$$\mathcal{L}\{e^{at} \cos(kt)\} = \frac{s-a}{(s-a)^2 + k^2} \quad , \quad \mathcal{L}\{e^{at} \sin(kt)\} = \frac{k}{(s-a)^2 + k^2}$$

$$X(s) = \frac{2/3}{s-1} + \frac{1/3(s-1)}{(s+\frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$Y(s) = \frac{2/3}{s-1} + \frac{1/3(s+2)}{(s+\frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

well use  $s-1 = s+\frac{1}{2} - \frac{3}{2}$  ,  $s+2 = s+\frac{1}{2} + \frac{3}{2}$

$$X(s) = \frac{2/3}{s-1} + \frac{1}{3} \frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2} - \frac{1}{3} \frac{3/2}{(s+\frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \cdot \frac{\frac{\sqrt{3}}{2}}{\frac{\sqrt{3}}{2}}$$

$$Y(s) = \frac{2/3}{s-1} + \frac{1}{3} \frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + \frac{1}{3} \frac{3/2}{(s+\frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \frac{\sqrt{3}/2}{\sqrt{3}/2}$$

$$\frac{1}{3} \cdot \frac{3}{2} \div \frac{\sqrt{3}}{2} = \frac{1}{3} \cdot \frac{3}{2} \cdot \frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$X(s) = \frac{2/3}{s-1} + \frac{1}{3} \frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} - \frac{1}{\sqrt{3}} \frac{\frac{\sqrt{3}}{2}}{(s+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2}$$

$$Y(s) = \frac{2/3}{s-1} + \frac{1}{3} \frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} + \frac{1}{\sqrt{3}} \frac{\frac{\sqrt{3}}{2}}{(s+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2}$$

The solution to the system is

$$x(t) = \frac{2}{3} e^t + \frac{1}{3} e^{\frac{1}{2}t} \cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{1}{\sqrt{3}} e^{\frac{1}{2}t} \sin\left(\frac{\sqrt{3}}{2}t\right)$$

$$y(t) = \frac{2}{3} e^t + \frac{1}{3} e^{\frac{1}{2}t} \cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{1}{\sqrt{3}} e^{\frac{1}{2}t} \sin\left(\frac{\sqrt{3}}{2}t\right)$$

## Section 17: Fourier Series: Trigonometric Series

Consider the following problem:

An undamped spring mass system has a mass of 2 kg attached to a spring with spring constant 128 N/m. The mass is driven by an external force  $f(t) = 2t$  for  $-1 < t < 1$  that is 2-periodic so that  $f(t + 2) = f(t)$  for all  $t > 0$ .

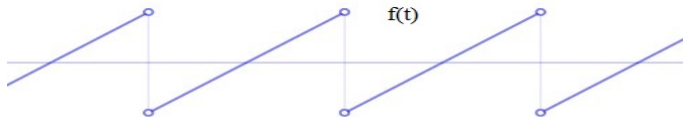
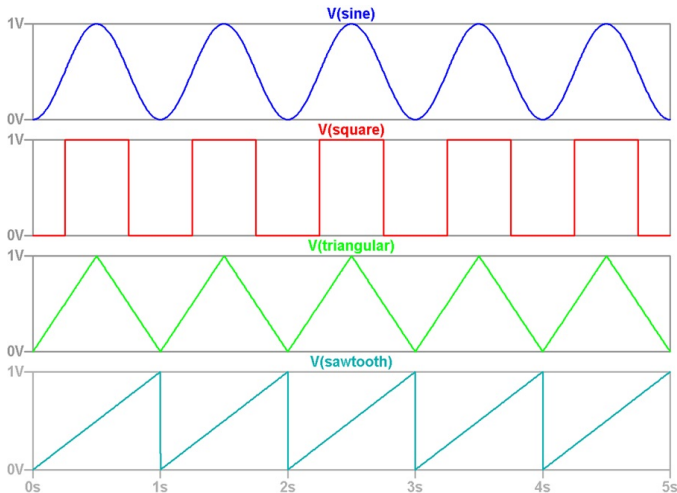


Figure:  $2 \frac{d^2x}{dt^2} + 128x = f(t)$

# Common Models of Periodic Sources (e.g. Voltage)



**Figure:** We'd like to solve, or at least approximate solutions, to ODEs and PDEs with periodic *right hand sides*.

# Series Representations for Functions

The goal is to represent a function by a series

$$f(x) = \sum_{n=1}^{\infty} (\text{some simple functions})$$

In calculus, you saw power series  $f(x) = \sum_{n=0}^{\infty} a_n(x - c)^n$  where the simple functions were powers  $(x - c)^n$ .

Here, you will see how some functions can be written as series of trigonometric functions

$$f(x) = \sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx)$$

We'll move the  $n = 0$  to the front before the rest of the sum.

## Some Preliminary Concepts

Suppose two functions  $f$  and  $g$  are integrable on the interval  $[a, b]$ . We define the **inner product** of  $f$  and  $g$  on  $[a, b]$  as

the inner  
product of  
 $f$  and  $g$   
on  $[a, b]$

$$\rightarrow \langle f, g \rangle = \int_a^b f(x)g(x) dx.$$

We say that  $f$  and  $g$  are **orthogonal** on  $[a, b]$  if

$$\langle f, g \rangle = 0.$$

The product depends on the interval, so the orthogonality of two functions depends on the interval.

# Properties of an Inner Product

Let  $f$ ,  $g$ , and  $h$  be integrable functions on the appropriate interval and let  $c$  be any real number. The following hold

$$(i) \quad \langle f, g \rangle = \langle g, f \rangle$$

$$(ii) \quad \langle f, g + h \rangle = \langle f, g \rangle + \langle f, h \rangle$$

$$(iii) \quad \langle cf, g \rangle = c \langle f, g \rangle$$

$$(iv) \quad \langle f, f \rangle \geq 0 \text{ and } \langle f, f \rangle = 0 \text{ if and only if } f = 0$$

$$\langle f, f \rangle = \int_a^b (f(x))^2 dx$$



## Orthogonal Set

A set of functions  $\{\phi_0(x), \phi_1(x), \phi_2(x), \dots\}$  is said to be **orthogonal** on an interval  $[a, b]$  if

$$\langle \phi_m, \phi_n \rangle = \int_a^b \phi_m(x) \phi_n(x) dx = 0 \quad \text{whenever} \quad m \neq n.$$

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Note that any function  $\phi(x)$  that is not identically zero will satisfy

$$\langle \phi, \phi \rangle = \int_a^b \phi^2(x) dx > 0.$$

Hence we define the **square norm** of  $\phi$  (on  $[a, b]$ ) to be

$$\|\phi\| = \sqrt{\int_a^b \phi^2(x) dx}.$$

# An Orthogonal Set of Functions

Consider the set of functions

$$\{1, \cos x, \cos 2x, \cos 3x, \dots, \sin x, \sin 2x, \sin 3x, \dots\} \quad \text{on} \quad [-\pi, \pi].$$

Evaluate  $\langle \cos(nx), 1 \rangle$  and  $\langle \sin(mx), 1 \rangle$ .

$[a, b]$

$$\begin{aligned}\langle \cos(nx), 1 \rangle &= \int_{-\pi}^{\pi} \cos(nx) \cdot 1 \, dx \\ &= \int_{-\pi}^{\pi} \cos(nx) \, dx \\ &= \frac{1}{n} \sin(nx) \Big|_{-\pi}^{\pi} \\ &= \frac{1}{n} \sin(n\pi) - \frac{1}{n} \sin(-n\pi) \\ &= \frac{1}{n} \cdot 0 - \frac{1}{n} \cdot 0 = 0\end{aligned}$$

For any integer  
 $n$   
 $\sin(n\pi) = 0$

Hence  $\cos(nx)$  is orthogonal to 1 on  $[-\pi, \pi]$ .

$$\begin{aligned}\langle \sin(mx), 1 \rangle &= \int_{-\pi}^{\pi} \sin(mx) \cdot 1 dx \\ &= \left. -\frac{1}{m} \cos(mx) \right|_{-\pi}^{\pi} \\ &= -\frac{1}{m} \cos(m\pi) - \left( -\frac{1}{m} \cos(-m\pi) \right) \\ &= -\frac{1}{m} \cos(m\pi) + \frac{1}{m} \cos(m\pi) \\ &= 0\end{aligned}$$

$\cos(-\theta) = \cos \theta$   
it's  
even

Hence  $\sin(mx)$  is orthogonal to 1 for all  $m$  on  $[-\pi, \pi]$ .

# An Orthogonal Set of Functions

Consider the set of functions

$$\{1, \cos x, \cos 2x, \cos 3x, \dots, \sin x, \sin 2x, \sin 3x, \dots\} \quad \text{on} \quad [-\pi, \pi].$$

It can easily be verified that

$$\int_{-\pi}^{\pi} \cos nx \, dx = 0 \quad \text{and} \quad \int_{-\pi}^{\pi} \sin mx \, dx = 0 \quad \text{for all} \quad n, m \geq 1,$$

$$\int_{-\pi}^{\pi} \cos nx \sin mx \, dx = 0 \quad \text{for all} \quad m, n \geq 1, \quad \text{and}$$

$$\int_{-\pi}^{\pi} \cos nx \cos mx \, dx = \int_{-\pi}^{\pi} \sin nx \sin mx \, dx = \begin{cases} 0, & m \neq n \\ \pi, & n = m \end{cases},$$

# An Orthogonal Set of Functions on $[-\pi, \pi]$

These integral values indicated that the set of functions

$$\{1, \cos x, \cos 2x, \cos 3x, \dots, \sin x, \sin 2x, \sin 3x, \dots\}$$

is an orthogonal set on the interval  $[-\pi, \pi]$ .

**Key Point:** This means that if we take any two functions  $f$  and  $g$  **from this set**, then

$$\int_{-\pi}^{\pi} f(x)g(x) dx = 0 \quad \text{if } f \text{ and } g \text{ are different functions!}$$

# Fourier Series

Suppose  $f(x)$  is defined for  $-\pi < x < \pi$ . We would like to know how to write  $f$  as a series **in terms of sines and cosines**.

**Task:** Find coefficients (numbers)  $a_0, a_1, a_2, \dots$  and  $b_1, b_2, \dots$  such that<sup>1</sup>

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

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<sup>1</sup>We'll write  $\frac{a_0}{2}$  as opposed to  $a_0$  purely for convenience.

# Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

The question of convergence naturally arises when we wish to work with infinite series. To highlight convergence considerations, some authors prefer not to use the equal sign when expressing a Fourier series and instead write

$$f(x) \sim \frac{a_0}{2} + \dots$$

Herein, we'll use the equal sign with the understanding that equality may not hold at each point.

Convergence will be address later.

## Finding an Example Coefficient

Let's find the coefficient  $b_4$ .

Start with the series  $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ , and multiply both sides by  $\sin(4x)$ .

$$f(x) \sin(4x) = \frac{a_0}{2} \sin(4x) + \sum_{n=1}^{\infty} (a_n \cos nx \sin(4x) + b_n \sin nx \sin(4x)).$$

Let's integrate from  $-\pi$  to  $\pi$

$$\int_{-\pi}^{\pi} f(x) \sin(4x) dx = \int_{-\pi}^{\pi} \frac{a_0}{2} \sin(4x) dx + \sum_{n=1}^{\infty} \left[ \int_{-\pi}^{\pi} a_n \cos(nx) \sin(4x) dx + \int_{-\pi}^{\pi} b_n \sin(nx) \sin(4x) dx \right]$$



Note  $\int_{-\pi}^{\pi} \frac{a_0}{2} \sin(4x) dx = \frac{a_0}{2} \int_{-\pi}^{\pi} \sin(4x) dx = 0$

$$\int_{-\pi}^{\pi} a_n \cos(nx) \sin(4x) dx = a_n \int_{-\pi}^{\pi} \cos(nx) \sin(4x) dx = 0$$

$$\int_{-\pi}^{\pi} b_n \sin(nx) \sin(4x) dx = b_n \int_{-\pi}^{\pi} \sin(nx) \sin(4x) dx$$

$$= 0 \quad \text{if } n \neq 4$$

But when  $n=4$   $b_4 \int_{-\pi}^{\pi} \sin(4x) \sin(4x) dx = b_4 \pi$

we end up with

$$\int_{-\pi}^{\pi} f(x) \sin(4x) dx = b_4 \pi$$

$$\Rightarrow b_4 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(4x) dx$$