November 10 Math 2306 sec. 54 Fall 2021

Section 16: Laplace Transforms of Derivatives and IVPs

Use the Laplace transform to solve the system of equations

$$\begin{aligned} x''(t) &= y, \quad x(0)=1, \quad x'(0)=0 \\ y'(t) &= x, \quad y(0)=1 \end{aligned}$$

We took the transform and used Crammer's Rule to get to the solution

$$\begin{aligned} X(s) &= \frac{2/3}{s-1} + \frac{1/3(s-1)}{s^2+s+1} \\ Y(s) &= \frac{2/3}{s-1} + \frac{1/3(s+2)}{s^2+s+1} \end{aligned}$$

$\frac{k}{s^2+k^2}$ or $\frac{s}{s^2+k^2}$

The irreducible quadratic denominator

$$s^2+s+1 = \left(s+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2.$$

$\mathcal{L}\{e^{at}\cos(kt)\} = \frac{s-a}{(s-a)^2+k^2}$, $\mathcal{L}\{e^{at}\sin(kt)\} = \frac{k}{(s-a)^2+k^2}$

$$X(s) = \frac{2/3}{s-1} + \frac{1/3(s-1)}{\left(s+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$Y(s) = \frac{2/3}{s-1} + \frac{1/3(s+2)}{\left(s+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

Note $s-1 = s+\frac{1}{2}-\frac{3}{2}$, $s+2 = s+\frac{1}{2}+\frac{3}{2}$

$$X(s) = \frac{2/3}{s-1} + \frac{1}{3}\,\frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2} - \frac{1}{3}\,\frac{3/2}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}\cdot\frac{\frac{\sqrt{3}}{2}}{\frac{\sqrt{3}}{2}}$$

$$Y(s) = \frac{2/3}{s-1} + \frac{1}{3}\,\frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2} + \frac{1}{3}\,\frac{\frac{3}{2}}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}\,\frac{\frac{\sqrt{3}}{2}}{\sqrt{3}/2}$$

$$\frac{1}{3}\cdot\frac{3}{2} \div \frac{\sqrt{3}}{2} = \frac{1}{3}\cdot\frac{3}{2}\cdot\frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$X(s) = \frac{2/3}{s-1} + \frac{1}{3}\,\frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+\left(\frac{\sqrt{3}}{2}\right)^2} - \frac{1}{\sqrt{3}}\,\frac{\frac{\sqrt{3}}{2}}{(s+\frac{1}{2})^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$Y(s) = \frac{2/3}{s-1} + \frac{1}{3}\,\frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+\left(\frac{\sqrt{3}}{2}\right)^2} + \frac{1}{\sqrt{3}}\,\frac{\frac{\sqrt{3}}{2}}{(s+\frac{1}{2})^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

The solution to the system is

$$x(t) = \frac{2}{3}e^{t} + \frac{1}{3}e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{1}{\sqrt{3}}e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{2}t\right)$$

$$y(t) = \frac{2}{3}e^{t} + \frac{1}{3}e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{1}{\sqrt{3}}e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{2}t\right)$$

Section 17: Fourier Series: Trigonometric Series

Consider the following problem:

An undamped spring mass system has a mass of 2 kg attached to a spring with spring constant 128 N/m. The mass is driven by an external force $f(t) = 2t$ for $-1 < t < 1$ that is 2-periodic so that $f(t+2) = f(t)$ for all $t > 0$.

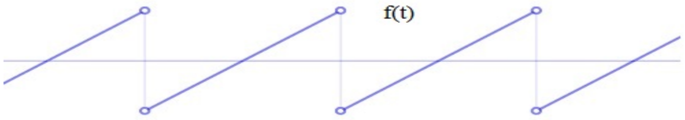

Figure: $$2\frac{d^2x}{dt^2} + 128x = f(t)$$

Common Models of Periodic Sources (e.g. Voltage)

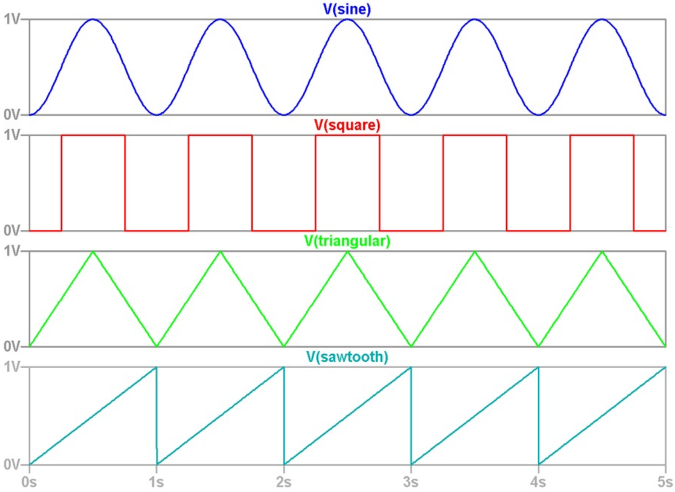

Figure: We'd like to solve, or at least approximate solutions, to ODEs and PDEs with periodic *right hand sides*.

Series Representations for Functions

The goal is to represent a function by a series

$$f(x) = \sum_{n=1}^{\infty} (\text{some simple functions})$$

In calculus, you saw power series $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ where the simple functions were powers $(x-c)^n$.

Here, you will see how some functions can be written as series of trigonometric functions

$$f(x) = \sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx)$$

We'll move the $n = 0$ to the front before the rest of the sum.

Some Preliminary Concepts

Suppose two functions f and g are integrable on the interval $[a, b]$. We define the **inner product** of f and g on $[a, b]$ as

the inner product of f and g on [a,b].

$$< f, g >= \int_a^b f(x)g(x)\,dx.$$

We say that f and g are **orthogonal** on $[a, b]$ if

$$< f, g >= 0.$$

The product depends on the interval, so the orthogonality of two functions depends on the interval.

Properties of an Inner Product

Let f, g, and h be integrable functions on the appropriate interval and let c be any real number. The following hold

(i) $< f, g >=< g, f >$

(ii) $< f, g + h >=< f, g > + < f, h >$

(iii) $< cf, g >= c < f, g >$

(iv) $< f, f >\geq 0$ and $< f, f >= 0$ if and only if $f = 0$

$$\langle f, f \rangle = \int_a^b (f(x))^2 \, dx$$

Orthogonal Set

A set of functions $\{\phi_0(x), \phi_1(x), \phi_2(x), \ldots\}$ is said to be **orthogonal** on an interval $[a, b]$ if

$$< \phi_m, \phi_n >= \int_a^b \phi_m(x)\phi_n(x)\, dx = 0 \quad \text{whenever} \quad m \neq n.$$

Note that any function $\phi(x)$ that is not identically zero will satisfy

$$< \phi, \phi >= \int_a^b \phi^2(x)\, dx > 0.$$

Hence we define the **square norm** of ϕ (on $[a, b]$) to be

$$\|\phi\| = \sqrt{\int_a^b \phi^2(x)\, dx}.$$

An Orthogonal Set of Functions

Consider the set of functions

$$\{1, \cos x, \cos 2x, \cos 3x, \ldots, \sin x, \sin 2x, \sin 3x, \ldots\} \quad \text{on} \quad [-\pi, \pi].$$

$[a, b]$

Evaluate $\langle \cos(nx), 1\rangle$ and $\langle \sin(mx), 1\rangle$.

$$\langle \cos(nx), 1\rangle = \int_{-\pi}^{\pi} \cos(nx)\cdot 1\, dx$$

$$= \frac{1}{n}\sin(nx)\Big|_{-\pi}^{\pi}$$

$$= \frac{1}{n}\sin(n\pi) - \frac{1}{n}\sin(-n\pi)$$

$$= \frac{1}{n}\cdot 0 - \frac{1}{n}\cdot 0 \quad = 0$$

for any integer n $\sin(n\pi) = 0$

Hence $\cos(nx)$ and 1 are orthogonal on $[-\pi, \pi]$.

$$\langle \sin(mx), 1\rangle = \int_{-\pi}^{\pi} \sin(mx)\cdot 1\, dx$$

$$= \frac{-1}{m}\cos(mx)\Big|_{-\pi}^{\pi}$$

$$= \frac{-1}{m}\cos(m\pi) - \frac{-1}{m}\cos(-m\pi)$$

$$= \frac{-1}{m}\cos(m\pi) + \frac{1}{m}\cos(m\pi)$$

$$= 0$$

$\cos(-\theta) = \cos\theta$

$\therefore$ its even

So $\sin(mx)$ and 1 are orthogonal on $[-\pi, \pi]$.

An Orthogonal Set of Functions

Consider the set of functions

$$\{1, \cos x, \cos 2x, \cos 3x, \ldots, \sin x, \sin 2x, \sin 3x, \ldots\} \quad \text{on} \quad [-\pi, \pi].$$

It can easily be verified that

$$\int_{-\pi}^{\pi} \cos nx\, dx = 0 \quad \text{and} \quad \int_{-\pi}^{\pi} \sin mx\, dx = 0 \quad \text{for all} \quad n, m \geq 1,$$

$$\int_{-\pi}^{\pi} \cos nx\, \sin mx\, dx = 0 \quad \text{for all} \quad m, n \geq 1, \quad \text{and}$$

$$\int_{-\pi}^{\pi} \cos nx\, \cos mx\, dx = \int_{-\pi}^{\pi} \sin nx\, \sin mx\, dx = \begin{cases} 0, & m \neq n \\ \pi, & n = m \end{cases},$$

An Orthogonal Set of Functions on $[-\pi, \pi]$

These integral values indicated that the set of functions

$$\{1, \cos x, \cos 2x, \cos 3x, \ldots, \sin x, \sin 2x, \sin 3x, \ldots\}$$

is an orthogonal set on the interval $[-\pi, \pi]$.

Key Point: This means that if we take any two functions f and g **from this set**, then

$$\int_{-\pi}^{\pi} f(x)g(x)\,dx = 0 \quad \text{if } f \text{ and } g \text{ are different functions!}$$

Fourier Series

Suppose $f(x)$ is defined for $-\pi < x < \pi$. We would like to know how to write f as a series **in terms of sines and cosines.**

Task: Find coefficients (numbers) $a_0, a_1, a_2, \dots$ and $b_1, b_2, \dots$ such that[1]

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right).$$

[1] We'll write $\frac{a_0}{2}$ as opposed to a_0 purely for convenience.

Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right).$$

The question of convergence naturally arises when we wish to work with infinite series. To highlight convergence considerations, some authors prefer not to use the equal sign when expressing a Fourier series and instead write

$$f(x) \sim \frac{a_0}{2} + \cdots$$

Herein, we'll use the equal sign with the understanding that equality may not hold at each point.

Convergence will be address later.

Finding an Example Coefficient

Let's find the coefficient b_4.

Start with the series $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$, and multiply both sides by $\sin(4x)$.

b_4 is coef of $\sin(4x)$

$$f(x)\sin(4x) = \frac{a_0}{2}\sin(4x) + \sum_{n=1}^{\infty}(a_n \cos nx \sin(4x) + b_n \sin nx \sin(4x)).$$

Now integrate both sides from $-\pi$ to π

$$\int_{-\pi}^{\pi} f(x)\sin(4x)\,dx = \int_{-\pi}^{\pi}\frac{a_0}{2}\sin(4x)\,dx + \sum_{n=1}^{\infty}\left[\int_{-\pi}^{\pi} a_n\cos(nx)\sin(4x)\,dx + \int_{-\pi}^{\pi} b_n\sin(nx)\sin(4x)\,dx\right]$$

Note $\int_{-\pi}^{\pi} \frac{a_0}{2} \sin(4x)\,dx = \frac{a_0}{2} \int_{-\pi}^{\pi} \sin(4x)\,dx = 0$

$$\int_{-\pi}^{\pi} a_n \cos(nx)\sin(4x)\,dx = a_n \int_{-\pi}^{\pi} \cos(nx)\sin(4x)\,dx = 0$$

$$\int_{-\pi}^{\pi} f(x)\sin(4x)\,dx = \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin(nx)\sin(4x)\,dx$$

$$\int_{-\pi}^{\pi} \sin(nx)\sin(4x)\,dx = 0 \quad \text{If} \quad n \neq 4$$

$$\int_{-\pi}^{\pi} \sin(4x)\sin(4x)\,dx = \pi$$

we have

$$\int_{-\pi}^{\pi} f(x)\sin(4x)\,dx = b_4 \pi$$

$$\Rightarrow \quad b_4 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(4x)\,dx$$

Finding Fourier Coefficients

Note that there was nothing special about seeking the 4^{th} sine coefficient b_4. We could have just as easily sought b_m for any positive integer m. We would simply start by introducing the factor $\sin(mx)$.

Moreover, using the same orthogonality property, we could pick on the a's by starting with the factor $\cos(mx)$—including the constant term since $\cos(0 \cdot x) = 1$. The only minor difference we want to be aware of is that

$$\int_{-\pi}^{\pi} \cos^2(mx)\,dx = \begin{cases} 2\pi, & m = 0 \\ \pi, & m \geq 1 \end{cases}$$

Careful consideration of this sheds light on why it is conventional to take the constant to be $\frac{a_0}{2}$ as opposed to just a_0.

The Fourier Series of $f(x)$ on $(-\pi, \pi)$

The **Fourier series** of the function f defined on $(-\pi, \pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)\,.$$

Where

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx, \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad \text{and} \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \end{aligned}$$