November 10 Math 3260 sec. 53 Fall 2025

4.8 Working with Coordinate Vectors

Theorem

Suppose S is a subspace of a vector space V and $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_k\}$ is an ordered basis of S. Let $T = \{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_m\}$ be any set of vectors in S, and let $C_T = \left\{[\vec{x}_1]_{\mathcal{B}}, [\vec{x}_2]_{\mathcal{B}}, \dots, [\vec{x}_m]_{\mathcal{B}}\right\}$ be the set of vectors in R^k consisting of the coordinate vectors of the elements of T with respect to the basis $\mathcal{B}$. Then T is linearly independent in V if and only if C_T is linearly independent in R^k.

The power of this theorem is that it will allow us to translate a problem in some finite dimensional vector space to R^k. Then we can use tools, like row reduction, to bear.

Example

Let $S = \{p, q, r\}$, where $p(x) = 1 + 4x - 2x^2 + 3x^3$,

$$q(x) = 1 + 3x - 3x^2 + x^3, \quad \text{and} \quad r(x) = 2 + 4x - 8x^2 - 2x^3.$$

Determine whether S is linearly dependent or linearly independent in $\mathbb{P}_3$. If linearly dependent, find a linear dependence relation.

We could conside the equation

$$c_1 p(x) + c_2 q(x) + c_3 r(x) = z(x)$$

[$z(x) = 0$ for all real x

Instead, we'll use coordinate vectors.

Let's use the ordered basis

$$\mathcal{B} = \{1, x, x^2, x^3\}$$

$[p]_{\mathcal{B}} = \langle p_0, p_1, p_2, p_3 \rangle$ where

$$p(x) = p_0(1) + p_1 x + p_2 x^2 + p_3 x^3$$

Let $S = \{p, q, r\}$, where $p(x) = 1 + 4x - 2x^2 + 3x^3$,

$q(x) = 1 + 3x - 3x^2 + x^3$, and $r(x) = 2 + 4x - 8x^2 - 2x^3$.

$[p]_{\mathcal{B}} = \langle 1, 4, -2, 3 \rangle$ $\qquad$ $[r]_{\mathcal{B}} = \langle 2, 4, -8, -2 \rangle$

$[q]_{\mathcal{B}} = \langle 1, 3, -3, 1 \rangle$

Let A have columns $[p]_{\mathcal{B}}, [q]_{\mathcal{B}}, [r]_{\mathcal{B}}$

then we can consider $A\vec{c} = \vec{0}_4$ $\quad$ $\vec{c} = \langle c_1, c_2, c_3 \rangle$

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 4 & 3 & 4 \\ -2 & -3 & -8 \\ 3 & 1 & -2 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The columns of A are linearly dependent.

So $\{p, q, r\}$ is lin. dependent in $\mathbb{P}_3$.

From the rref $[r]_{\mathcal{B}} = -2[p]_{\mathcal{B}} + 4[q]_{\mathcal{B}}$

Let $S = \{p, q, r\}$, where $p(x) = 1 + 4x - 2x^2 + 3x^3$,

$$q(x) = 1 + 3x - 3x^2 + x^3, \quad \text{and} \quad r(x) = 2 + 4x - 8x^2 - 2x^3.$$

So $r(x) = -2p(x) + 4q(x)$

Verify: $-2p(x) + 4q(x) = -2 - 8x + 4x^2 - 6x^3 + 4 + 12x - 12x^2 + 4x^3$

$= 2 + 4x - 8x^2 - 2x^3$

A linear dependence relation for $\{p, q, r\}$ is $2p(x) - 4q(x) + r(x) = z(x)$.

Chapter 5 Linear Transformations

In this chapter, we will consider a special class of functions called **linear transformations.** The inputs and outputs that we'll be interested in will be vectors.

Let's start with some notation and concepts related to functions more generally.

Domain, Codomain, Images, & Range

"f maps D into C"

$$\underbrace{f}_{\text{Rule}} : \underbrace{D}_{\text{set of all inputs}} \longrightarrow \underbrace{C}_{\text{set where outputs live}}$$

- D is the **domain** of the function.
- C is where the outputs are. It's called the **codomain**.
- For x in D, if $f(x) = y$, we call y the **image of x under f**.
- If S is a subset of D, then the **image of S under f** is the collection of all images for each x in S.

$$f(S) = \{y \in C \mid y = f(x) \text{ for at least one } x \in S\} = \{f(x) \mid x \in S\}$$

- $f(D)$ is the **range** of f. This is the set of all actual images. We can write $f(D) = \mathrm{range}(f)$.

Codomain -vs- Range

Codomain = the set that contains the outputs

Range = the set of actual outputs

Example: Consider $f : R \to R$ defined by $f(x) = e^x$. The **codomain** is R because that's how the function is being defined. But recall that

$$e^x > 0, \quad \text{for all real } x.$$

So the **range** is the interval $(0, \infty)$.

Example: Consider $g : (0, \infty) \to R$ defined by $g(x) = \ln(x)$. For this function

$$\text{the } \textbf{codomain} = R = \text{the } \textbf{range}.$$

Onto, One-to-One, & Invertibility

$$f : D \to C$$

f maps D into C

Onto

If $f(D) = C$, that is, if the range is equal to the codomain, we say that f is **onto.** In this case, we say

"f maps D **onto** C."

If f maps D onto C, then for each $y \in C$

$$f(x) = y \quad \text{is consistent.}$$

By consistent, we mean has at least one solution.

Onto, One-to-One, & Invertibility

$$f : D \to C$$

One-to-One

If for each $y \in \text{range}(f)$, the equation

$$f(x) = y$$

has exactly one solution, we say that f is **one-to-one.**

$f(x) = f(z)$ is only true if $x = z$.

If f is one-to-one, then for each y such that $f(x) = y$ is consistent

$$f(x) = y \quad \text{has a unique solution.}$$

Onto, One-to-One, & Invertibility

$$f : D \to C$$

Invertible

If f maps D onto C and f is one-to-one, then we say that f is **invertible**. If $f : D \to C$ is invertible, then there is a corresponding **inverse function** denoted by f^{-1} such that

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x, \quad \text{for each } x \in D, \text{ and}$$

$$(f \circ f^{-1})(x) = f(f^{-1}(x)) = x, \quad \text{for each } x \in C.$$

$f^{-1} : C \to D$ is the function defined by

$$f^{-1}(y) = x \text{ where } x \text{ is the unique solution of } f(x) = y.$$

Example: Let $T : R^2 \to R^2$ be defined by $T(\vec{x}) = A\vec{x}$ where $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

1. Find the images of $\vec{x} = \langle 1, 1 \rangle$, $\vec{x} = \langle -2, 1 \rangle$, and $\vec{x} = \langle x_1, x_2 \rangle$.

$$\underset{m\times n}{A}\ \underset{R^n}{\vec{x}} = \underset{R^m}{\vec{y}}$$

$$T(\langle 1,1\rangle) = A\langle 1,1\rangle = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\langle 1,1\rangle$$

$$= \langle 0(1)+(-1)(1),\ 1(1)+0(1)\rangle$$

$$= \langle -1, 1\rangle$$

$$T(\langle -2,1\rangle) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\langle -2,1\rangle = \langle -1,-2\rangle$$

$$T(\langle x_1,x_2\rangle) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\langle x_1,x_2\rangle = \langle -x_2, x_1\rangle.$$

Example: Let $T : R^2 \to R^2$ be defined by $T(\vec{x}) = A\vec{x}$ where $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

1. Find the images of $\vec{x} = \langle 1, 1 \rangle$, $\vec{x} = \langle -2, 1 \rangle$, and $\vec{x} = \langle x_1, x_2 \rangle$.
2. What is $\mathrm{range}(T)$? Does T map R^2 onto R^2?

range $(T) = \{ \vec{y} \in R^2 \mid T(\vec{x}) = \vec{y}$ is consistent$\}$.

For what $\vec{y}$ is $A\vec{x} = \vec{y}$ consistent?

$A \xrightarrow{\text{rref}} I_2$. A has a pivot position in both rows. $A\vec{x} = \vec{y}$ is always consistent. range $(T) = R^2$ and T is onto.

Example: Let $T : R^2 \to R^2$ be defined by $T(\vec{x}) = A\vec{x}$ where $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

1. Find the images of $\vec{x} = \langle 1, 1 \rangle$, $\vec{x} = \langle -2, 1 \rangle$, and $\vec{x} = \langle x_1, x_2 \rangle$.
2. What is $\mathrm{range}(T)$? Does T map R^2 onto R^2?
3. Is T one-to-one?

If $T(\vec{x}) = T(\vec{z})$ does $\vec{x}$ have to equal $\vec{z}$? When $A\vec{x} = \vec{y}$ is consistent, is the solution unique?

Since $\mathrm{rref}(A) = I_2$, both columns are pivot columns. So $A\vec{x} = \vec{y}$ always has a unique solution.

T is one to one.

Example: Let $T : R^2 \to R^2$ be defined by $T(\vec{x}) = A\vec{x}$ where $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

1. Find the images of $\vec{x} = \langle 1, 1 \rangle$, $\vec{x} = \langle -2, 1 \rangle$, and $\vec{x} = \langle x_1, x_2 \rangle$.
2. What is $\mathrm{range}(T)$? Does T map R^2 onto R^2?
3. Is T one-to-one?
4. Is T invertible?

T is invertible since T is onto and one to one.

Example: Let $P : R^2 \to R^2$ be defined by $P(\vec{x}) = B\vec{x}$ where $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

1. Find the images of $\vec{x} = \langle 1, 1 \rangle$, $\vec{x} = \langle -2, 1 \rangle$, and $\vec{x} = \langle x_1, x_2 \rangle$.

$$P(\langle 1,1\rangle) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \langle 1,1\rangle = \langle 0, 1\rangle$$

$$P(\langle -2,1\rangle) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \langle -2,1\rangle = \langle 0, 1\rangle$$

$$P(\langle x_1,x_2\rangle) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \langle x_1,x_2\rangle = \langle 0, x_2\rangle$$

Example: Let $P : R^2 \to R^2$ be defined by $P(\vec{x}) = B\vec{x}$ where $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

1. Find the images of $\vec{x} = \langle 1, 1 \rangle$, $\vec{x} = \langle -2, 1 \rangle$, and $\vec{x} = \langle x_1, x_2 \rangle$.
2. What is $\mathrm{range}(P)$? Does P map R^2 onto R^2?

range $(P) = \text{Span}\{\langle 0, 1\rangle\}$.

Since every image $\vec{y} = \langle 0, y \rangle$ for some real y.

P isn't onto. No vector in R^2 with a nonzero first entry is in the range. For example $P(\langle x_1, x_2 \rangle) = \langle 1, 0 \rangle$ is not consistent.

Example: Let $P : R^2 \to R^2$ be defined by $P(\vec{x}) = B\vec{x}$ where $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

1. Find the images of $\vec{x} = \langle 1, 1 \rangle$, $\vec{x} = \langle -2, 1 \rangle$, and $\vec{x} = \langle x_1, x_2 \rangle$.
2. What is $\mathrm{range}(P)$? Does P map R^2 onto R^2?
3. Is P one-to-one?

$P(\langle 1, 1 \rangle) = P(\langle -2, 1 \rangle)$

P is not one to one

Two different inputs gave the same output.
That won't happen with a one to one function.

Example: Let $P : R^2 \to R^2$ be defined by $P(\vec{x}) = B\vec{x}$ where $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

1. Find the images of $\vec{x} = \langle 1, 1 \rangle$, $\vec{x} = \langle -2, 1 \rangle$, and $\vec{x} = \langle x_1, x_2 \rangle$.
2. What is $\mathrm{range}(P)$? Does P map R^2 onto R^2?
3. Is P one-to-one?
4. Is P invertible?

P is not invertible.

P is neither onto nor one to one, so it's not invertible.

5.2 Linear Transformations for R^n to R^m

Linear Transformation

A **linear transformation** from R^n to R^m is a function $T : R^n \to R^m$ such that for each pair of vectors $\vec{x}$ and $\vec{y}$ in R^n and for any scalar c

1. $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$, and
2. $T(c\vec{x}) = cT(\vec{x})$.

A function having vector spaces as a domain and codomain are called **transformations.**

The two properties in this definition are what we mean by **linear** or **linearity**. Functions that don't have these properties are called nonlinear.

Example

Let $T : R^2 \to R^3$ be defined by $T(\langle x_1, x_2 \rangle) = \langle x_1 x_2, 0, x_1 + x_2 \rangle$.

Find the images of $\langle 0, 0 \rangle$, $\langle 1, 0 \rangle$, $\langle 0, 1 \rangle$, $\langle 1, 1 \rangle$ and $\langle 2, 2 \rangle$.

1. $T(\langle 0, 0 \rangle) =$

2. $T(\langle 1, 0 \rangle) =$

3. $T(\langle 0, 1 \rangle) =$

4. $T(\langle 1, 1 \rangle) =$

5. $T(\langle 2, 2 \rangle) =$

$T : R^2 \to R^3 \quad T(\langle x_1, x_2 \rangle) = \langle x_1 x_2, 0, x_1 + x_2 \rangle$

1. Is $T(\langle 1, 0 \rangle + \langle 0, 1 \rangle) = T(\langle 1, 0 \rangle) + T(\langle 0, 1 \rangle)$?

2. Is $T(2\langle 1, 1 \rangle) = 2T(\langle 1, 1 \rangle)$?

Example

Let $T : R^3 \to R^2$ be defined by $T(\langle x_1, x_2, x_3 \rangle) = \langle x_1 - x_2, x_2 - x_3 \rangle$.
Show that T is a linear transformation.

$T : R^3 \to R^2 \quad T(\langle x_1, x_2, x_3 \rangle) = \langle x_1 - x_2, x_2 - x_3 \rangle$

Recall Properties of Matrix-Vector Product

If A is an $m \times n$ matrix, $\vec{x}$ and $\vec{y}$ are vectors in R^n, and c is a scalar, then

- $A\vec{x}$ is a vector in R^m.
- $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$, and
- $A(c\vec{x}) = cA\vec{x}$.

Lemma

If A is an $m \times n$ matrix and $T : R^n \to R^m$ is defined by $T(\vec{x}) = A\vec{x}$, then T is a linear transformation.

Not only does the matrix-vector product define a linear transformation. Turns out, **every** linear transformation from R^n to R^m is a matrix-vector product!

Theorem

Suppose that $T : R^n \to R^m$ is a linear transformation. Then there is a unique $m \times n$ matrix A, such that $T(\vec{x}) = A\vec{x}$ for all $\vec{x} \in R^n$.

Furthermore, the matrix[a] A is the matrix whose column vectors are

$$\mathrm{Col}_j(A) = T\left(\vec{e}_j\right)$$

where $\mathcal{E} = \left\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\right\}$ is the standard basis for R^n.

[a] We'll call A the **standard matrix** for the transformation T.

The **columns** of A are the images of the standard unit vectors.

A is unique if we are considering inputs and outputs relative to the standard basis $\mathcal{E}$.

Example

Find the standard matrix for the linear transformation $T : R^2 \to R^3$ given by

$$T(\langle x_1, x_2 \rangle) = \langle x_1 + 3x_2, 2x_1 + 4x_2, -2x_2 \rangle.$$

Theorem

If $T : R^n \to R^m$ is a linear transformation, then

$$T(\vec{0}_n) = \vec{0}_m.$$

Remark: This can be used as a test to rule out that something is a linear transformation. That is, if for some $T : R^n \to R^m$, $T(\vec{0}_n) \neq \vec{0}_m$, then T can't be a linear transformation.

Caveat: This doesn't say that $T(\vec{0}_n) = \vec{0}_m$ by itself guarantees linearity.

Fundamental Subspaces: Range and Kernel

Let $T : R^n \to R^m$ be a linear transformation, and let A be its standard matrix. The **range** of T is defined by

$$\mathrm{range}(T) = \{T(\vec{x}) \mid \vec{x} \in R^n\}.$$

and the **kernel** of T, denoted $\ker(T)$ is defined by

$$\ker(T) = \left\{\vec{x} \in R^n \mid T(\vec{x}) = \vec{0}_m\right\}.$$

Moreover, $\mathrm{range}(T)$ is a subspace of R^m, $\ker(T)$ is a subspace of R^n, and

$$\mathrm{range}(T) = \mathcal{CS}(A), \quad \text{and} \quad \ker(T) = \mathcal{N}(A).$$

It follows from the FTLA that

$$\dim\left(\mathrm{range}(T)\right) + \dim\left(\ker(T)\right) = n.$$

Example

Identify the range and kernel of $T : R^3 \to R^2$ given by

$$T(\langle x_1, x_2, x_3 \rangle) = \langle 2x_1 + x_2 - 2x_3, 2x_1 + 3x_2 + 6x_3 \rangle.$$

$$T(\langle x_1, x_2, x_3\rangle) = \langle 2x_1 + x_2 - 2x_3, 2x_1 + 3x_2 + 6x_3\rangle.$$

$T(\langle x_1, x_2, x_3\rangle) = \langle 2x_1 + x_2 - 2x_3, 2x_1 + 3x_2 + 6x_3\rangle.$

1. Is T onto?

2. Is T one to one?

3. Is T invertible?

Range & Kernel Dimensions

Onto & One to One Indicators

Theorem

Let $T : R^n \to R^m$ be a linear transformation. Then

1. T is onto if and only if $\dim(\text{range}(T)) = m$, and
2. T is one-to-one if and only if $\dim(\ker(T)) = 0$.

The second statement can be rephrased as saying that T is one-to-one if and only if

$$T(\vec{x}) = \vec{0}_m$$

has only the trivial solution, $\vec{x} = \vec{0}_n$.

Onto, One-to-One & Standard Matrix

Suppose $T : R^n \to R^m$ is a linear transformation with standard matrix A.

> Since $\text{range}(T) = \mathcal{CS}(A)$, T is onto if and only if A has a pivot in every row.

> Since $\ker(T) = \mathcal{N}(A)$, T is one-to-one if and only if all columns of A are pivot columns.

Note that since $\dim\big(\text{range}(T)\big) + \dim\big(\ker(T)\big) = n$, the only way for T to be both onto and one-to-one is for $m = n$. That is, $T : R^n \to R^n$ and A is a square matrix.

Invertible Linear Transformations

Inverse of a Linear Transformation

Let $T : R^n \rightarrow R^n$ be a linear transformation with standard matrix A. Then T is invertible if and only if A is an invertible matrix. In this case,

$$T^{-1}(\vec{x}) = A^{-1}\vec{x}$$

for each $\vec{x}$ in R^n.

The standard matrix for T^{-1} is the inverse of the standard matrix for T.