November 11 Math 2306 sec. 51 Fall 2022

Section 16: Laplace Transforms of Derivatives and IVPs

Transforms of Derivatives For $y = y(t)$ defined on $[0, \infty)$ having derivatives y', y'' and so forth, if

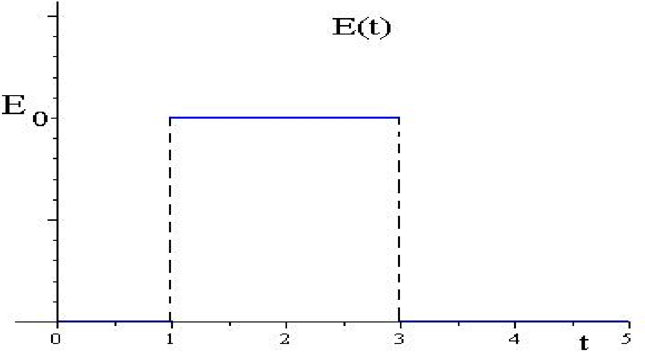
$$\mathscr{L}\{y(t)\} = Y(s),$$

then

$$\begin{aligned}
\mathscr{L}\left\{\frac{dy}{dt}\right\} &= sY(s) - y(0),\\
\mathscr{L}\left\{\frac{d^2y}{dt^2}\right\} &= s^2Y(s) - sy(0) - y'(0),\\
\mathscr{L}\left\{\frac{d^3y}{dt^3}\right\} &= s^3Y(s) - s^2y(0) - sy'(0) - y''(0),\\
&\vdots \qquad\qquad \vdots\\
\mathscr{L}\left\{\frac{d^ny}{dt^n}\right\} &= s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - y^{(n-1)}(0).
\end{aligned}$$

Solve the IVP

An LR-series circuit has inductance $L = 1$h, resistance $R = 10\Omega$, and applied force $E(t)$ whose graph is given below. If the initial current $i(0) = 0$, find the current $i(t)$ in the circuit.



LR Circuit Example

We determined that the IVP is

$$\frac{di}{dt}+10i=E_0\mathscr{U}(t-1)-E_0\mathscr{U}(t-3),\quad i(0)=0$$

Let $I(s)=\mathscr{L}\{i(t)\}$

$$\mathscr{L}\{i'+10i\}=\mathscr{L}\{E_0\mathscr{U}(t-1)-E_0\mathscr{U}(t-3)\}$$

$$\mathscr{L}\{i'\}+10\mathscr{L}\{i\}=E_0\mathscr{L}\{\mathscr{U}(t-1)\}-E_0\mathscr{L}\{\mathscr{U}(t-3)\}$$

$$sI(s)-\underbrace{i(0)}_{0}+10I(s)=E_0\frac{e^{-s}}{s}-E_0\frac{e^{-3s}}{s}$$

$$(s+10)I(s)=E_0\frac{e^{-s}}{s}-E_0\frac{e^{-3s}}{s}$$

$$I(s) = E_0 \frac{e^{-s}}{s(s+10)} - E_0 \frac{e^{-3s}}{s(s+10)}$$

Let's decompose $\dfrac{1}{s(s+10)}$

$$\frac{1}{s(s+10)} = \frac{A}{s} + \frac{B}{s+10} \Rightarrow$$

$$1 = A(s+10) + Bs$$

$s=0$ $\quad 1 = 10A \Rightarrow A = \frac{1}{10}$

$s=-10$ $\quad 1 = -10B \Rightarrow B = \frac{-1}{10}$

$$I(s) = \frac{E_0}{10} e^{-s}\left(\frac{1}{s} - \frac{1}{s+10}\right) - \frac{E_0}{10} e^{-3s}\left(\frac{1}{s} - \frac{1}{s+10}\right)$$

$$\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as}F(s)$$

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathcal{U}(t-a)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+10}\right\} = 1 - e^{-10t} = f(t)$$

$$I(s) = \frac{E_0}{10}e^{-s}\left(\frac{1}{s} - \frac{1}{s+10}\right) - \frac{E_0}{10}e^{-3s}\left(\frac{1}{s} - \frac{1}{s+10}\right)$$

The solution to the IVP $i(t) = \mathcal{L}^{-1}\{I(s)\}$

$$i(t) = \frac{E_0}{10}\left(1 - e^{-10(t-1)}\right)\mathcal{U}(t-1) - \frac{E_0}{10}\left(1 - e^{-10(t-3)}\right)\mathcal{U}(t-3)$$

We can rewrite i using stacked notation

$$i(t) = \begin{cases} 0, & 0 \le t < 1 \\ \dfrac{E_0}{10} - \dfrac{E_0}{10} e^{-10(t-1)}, & 1 \le t < 3 \\ \dfrac{E_0}{10} e^{-10(t-3)} - \dfrac{E_0}{10} e^{-10(t-1)}, & t \ge 3 \end{cases}$$

for $0 \le t < 1$ $\quad \mathcal{U}(t-1) = \mathcal{U}(t-3) = 0$

for $1 \le t < 3$ $\quad \mathcal{U}(t-1) = 1$ and $\mathcal{U}(t-3) = 0$

for $t \ge 3$ $\quad \mathcal{U}(t-1) = 1$ and $\mathcal{U}(t-3) = 1$

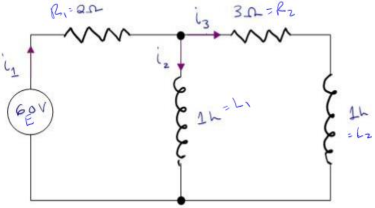
Solving a System

We can solve a system of ODEs using Laplace transforms. Here, we'll consider systems that are

- linear,
- having initial conditions at $t = 0$, and
- constant coefficient.

Let's see it in action (i.e. with a couple of examples).

Example



$$
\begin{aligned}
L_1 i_2' &+ R_1(i_2 + i_3) && = E \\
L_2 i_3' &+ R_1(i_2 + i_3) + R_2 i_3 && = E
\end{aligned}
$$

Figure: If we label current i_2 as $x(t)$ and current i_3 as $y(t)$, we get the system of equations below. (Assuming $i_1(0) = 0$.)

Solve the system of equations

$$
\begin{aligned}
\frac{dx}{dt} &= -2x - 2y + 60, \quad x(0) = 0 \\
\frac{dy}{dt} &= -2x - 5y + 60, \quad y(0) = 0
\end{aligned}
$$

$$\frac{dx}{dt} = -2x - 2y + 60, \quad x(0) = 0$$
$$\frac{dy}{dt} = -2x - 5y + 60, \quad y(0) = 0$$

Let $X(s) = \mathcal{L}\{x(t)\}$ and $Y(s) = \mathcal{L}\{y(t)\}$

$$\mathcal{L}\{x'\} = \mathcal{L}\{-2x - 2y + 60\}$$
$$= -2\mathcal{L}\{x\} - 2\mathcal{L}\{y\} + 60\mathcal{L}\{1\}$$

$$sX(s) - \underset{\overset{\shortparallel}{0}}{x(0)} = -2X(s) - 2Y(s) + \frac{60}{s}$$

$$\mathcal{L}\{y'\} = -2\mathcal{L}\{x\} - 5\mathcal{L}\{y\} + 60\mathcal{L}\{1\}$$

$$sY(s) - y(0) = -2X(s) - 5Y(s) + \frac{60}{s}$$

$$0$$

$$sX + 2X + 2Y = \frac{60}{s}$$

$$2X + sY + 5Y = \frac{60}{s}$$

$$(s+2)X + 2Y = \frac{60}{s}$$

$$2X + (s+5)Y = \frac{60}{s}$$

We'll pick this up next time using Cramer's rule.