November 11 Math 2306 sec. 53 Fall 2024

Section 15: Shift Theorems

Theorem Shift in s:

Suppose $\mathscr{L}\{f(t)\} = F(s)$. Then for any real number a

$$\mathscr{L}\left\{e^{at}f(t)\right\} = F(s-a).$$

From the perspective of the inverse transform

$$\mathscr{L}^{-1}\{F(s-a)\} = e^{at}\mathscr{L}^{-1}\{F(s)\}.$$

Inverse Laplace Transforms (repeated linear factors)

(b) $\mathscr{L}^{-1}\left\{\dfrac{1+3s-s^2}{s(s-1)^2}\right\}$

Do partial fraction decomp.

$$\frac{1+3s-s^2}{s(s-1)^2} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{(s-1)^2}$$

Omitting the details, $A = 1$, $B = -2$, $C = 3$

$$\mathscr{L}^{-1}\left\{\frac{1+3s-s^2}{s(s-1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{2}{s-1} + \frac{3}{(s-1)^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 2\,\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + 3\,\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\}$$

$$\frac{1}{s^2}$$

$$= 1 - 2e^{1t} + 3e^{1t}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$$

$$= 1 - 2e^{t} + 3te^{t}$$

The Unit Step Function

Definition: Unit Step Function

Let $a > 0$. The unit step function *centered at a* is denoted $\mathscr{U}(t-a)$. It is defined by

$$\mathscr{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a \end{cases}$$

The name *unit step function* is derived from its graph which looks like a stair step of height 1.

$\mathcal{U}(t) = 1$ for all $t \geq 0$.

Heaviside step

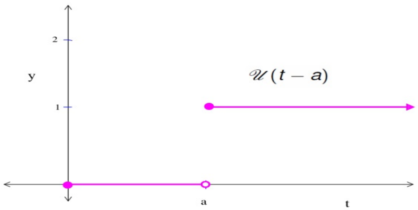

Figure: A graph of $\mathscr{U}(t-a)$ which jumps from zero to one at $t = a$.

Piecewise Defined Functions

Verify that

$$
\begin{aligned}
f(t) &= \begin{cases} g(t), & 0 \le t < a \\ h(t), & t \ge a \end{cases} \\
&= g(t) - g(t)\mathscr{U}(t-a) + h(t)\mathscr{U}(t-a)
\end{aligned}
$$

$$\mathscr{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a \end{cases}$$

To show equivalence, we have to consider $0 \le t < a$ and $t \ge a$. For $0 \le t < a$,

$\mathscr{U}(t-a) = 0$.

$g(t) - g(t)\mathscr{U}(t-a) + h(t)\mathscr{U}(t-a) = g(t) - g(t)(0) + h(t)(0) = g(t)$

$$\begin{cases} g(t), & 0 \le t < a \\ h(t), & t \ge a \end{cases} = g(t) - g(t)\mathscr{U}(t-a) + h(t)\mathscr{U}(t-a)$$

When $t \geq a$, $u(t-a) = 1$

$$g(t) - g(t)u(t-a) + h(t)u(t-a) =$$

$$g(t) - g(t)(1) + h(t)(1) =$$

$$g(t) - g(t) + h(t) = h(t)$$

$$\text{Example } f(t) = \begin{cases} e^t, & 0 \le t < 2 \\ t^2, & 2 \le t < 5 \\ 2t & t \ge 5 \end{cases}$$

Rewrite the function f in terms of the unit step function.

$$f(t) = e^t - e^t u(t-2) + t^2 u(t-2) - t^2 u(t-5) + 2t\, u(t-5)$$

could add $u(t-0) = u(t)$

could think of $-2t\, u(t-\infty)$ added

Find $\mathscr{L}\{\mathscr{U}(t-a)\}$

$$\mathscr{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a \end{cases}$$

$$\mathscr{L}\{\mathcal{U}(t-a)\} = \int_0^\infty e^{-st}\,\mathcal{U}(t-a)\,dt$$

$$= \int_0^a e^{-st}(0)\,dt + \int_a^\infty e^{-st}(1)\,dt$$

$$= \int_a^\infty e^{-st}\,dt$$ only convergent if $s>0$

for $s>0$

$$= \frac{-1}{s} e^{-st}\Big|_a^\infty = \frac{-1}{s}\left(0 - e^{-as}\right) = \frac{e^{-as}}{s}$$

Translation in t

Given a function $f(t)$ for $t \geq 0$, and a number $a > 0$

$$f(t-a)\mathscr{U}(t-a) = \begin{cases} 0, & 0 \leq t < a \\ f(t-a), & t \geq a \end{cases}.$$

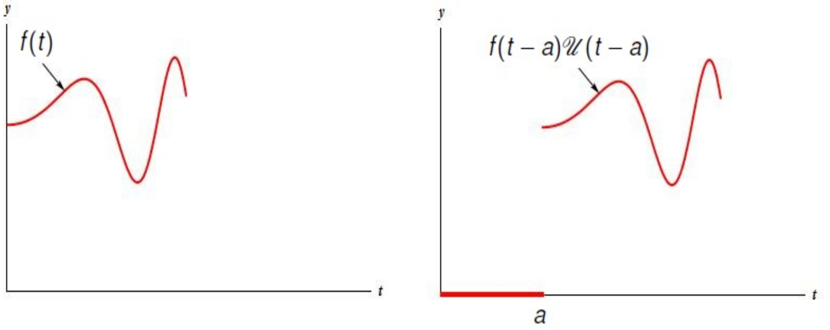

Figure: The function $f(t-a)\mathscr{U}(t-a)$ has the graph of f shifted a units to the right with value of zero for t to the left of a.

Theorem (translation in t)

If $F(s) = \mathscr{L}\{f(t)\}$ and $a > 0$, then

$$\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} = e^{-as}F(s).$$

A special case is $f(t) = 1$. We just found
$\mathscr{L}\{\mathscr{U}(t-a)\} = e^{-as}\mathscr{L}\{1\} = \dfrac{e^{-as}}{s}$.

We can state this in terms of the inverse transform as

$$\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathscr{U}(t-a).$$

Example

Find the Laplace transform $\mathscr{L}\{f(t)\}$ where

$$f(t) = \begin{cases} 1, & 0 \leq t < 1 \\ t, & t \geq 1 \end{cases}$$

(a) First write f in terms of unit step functions.

$$f(t) = 1 - 1\,\mathcal{U}(t-1) + t\,\mathcal{U}(t-1)$$

$$= 1 + (-1 + t)\,\mathcal{U}(t-1)$$

$$= 1 + (t-1)\,\mathcal{U}(t-1)$$

Example Continued...

(b) Now use the fact that $f(t) = 1 + (t-1)\mathscr{U}(t-1)$ to find $\mathscr{L}\{f\}$.

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{1 + (t-1)u(t-1)\}$$

$$= \mathcal{L}\{1\} + \mathcal{L}\{(t-1)u(t-1)\}$$

$$= \frac{1}{s} + e^{-1s}\mathcal{L}\{t\}$$

$$= \frac{1}{s} + e^{-s}\left(\frac{1}{s^2}\right) = \frac{1}{s} + \frac{e^{-s}}{s^2}$$

$(t-1)u(t-1) = g(t-1)u(t-1)$ If $g(t-1) = t-1$

$g(t) = t$

Alternative Form for Translation in t

It is often the case that we wish to take the transform of a product of the form

$$g(t)\mathscr{U}(t-a)$$

in which the function g is not translated.

The main theorem statement

$$\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} = e^{-as}F(s).$$

can be restated as

$$\boxed{\mathscr{L}\{g(t)\mathscr{U}(t-a)\} = e^{-as}\mathscr{L}\{g(t+a)\}.}$$

This is based on the observation that

$$g(t) = g((t+a)-a).$$