November 12 Math 2306 sec. 52 Fall 2021

Section 17: Fourier Series: Trigonometric Series

Consider a function $f(x)$ defined on the interval $[-\pi, \pi]$. A series of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)\,.$$

is called the **Fourier series** or **Trigonometric series** of f.

An Orthogonal Set of Functions on $[-\pi, \pi]$

Recall that the set of functions

$$\{1, \cos x, \cos 2x, \cos 3x, \ldots, \sin x, \sin 2x, \sin 3x, \ldots\}$$

is an orthogonal set on the interval $[-\pi, \pi]$.

Key Point: This means that if we take any two functions f and g **from this set**, then

$$\int_{-\pi}^{\pi} f(x)g(x)\,dx = 0 \quad \text{if } f \text{ and } g \text{ are different functions!}$$

Fourier Series

Suppose $f(x)$ is defined for $-\pi < x < \pi$. We would like to know how to write f as a series **in terms of sines and cosines.**

Task: Find coefficients (numbers) a_0, $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ such that[1]

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right).$$

[1] We'll write $\frac{a_0}{2}$ as opposed to a_0 purely for convenience.

Finding an Example Coefficient

Let's find the coefficient b_4.

Start with the series $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$, and multiply both sides by $\sin(4x)$.

$$f(x)\sin(4x) = \frac{a_0}{2}\sin(4x) + \sum_{n=1}^{\infty}\left(a_n \cos nx \sin(4x) + b_n \sin nx \sin(4x)\right).$$

We integrated from $-\pi$ to π. Using the orthogonality property, we came to the conclusion that

$$\int_{-\pi}^{\pi} f(x)\sin(4x)\,dx = \pi b_4$$

giving us a formula

$$b_4 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(4x)\,dx.$$

$$\frac{\langle f(x), \sin(4x)\rangle}{\langle \sin(4x), \sin(4x)\rangle}$$

Finding an Example Coefficient

Let's find the constant term $a_0/2$. We'll follow the same procedure. Start by assuming f has the series representation.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right).$$

- Multiply both sides of the equation by 1,
- integrate from $-\pi$ to π.

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} \frac{a_0}{2}\,dx + \sum_{n=1}^{\infty} \left[\int_{-\pi}^{\pi} a_n \cos(nx)\,dx + \int_{-\pi}^{\pi} b_n \sin(nx)\,dx \right]$$

By orthogonality

$$\int_{-\pi}^{\pi} \cos(nx)\,dx = 0 \quad \text{and} \quad \int_{-\pi}^{\pi} \sin(nx)\,dx = 0 \quad \text{for all } n = 1, 2, 3, \dots$$

Everything vanishes except

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} \frac{a_0}{2}\,dx = \frac{a_0}{2}\int_{-\pi}^{\pi} dx$$

$$= \frac{a_0}{2}\left[x\,\Big|_{-\pi}^{\pi}\right. = \frac{a_0}{2}\left[\pi - (-\pi)\right]$$

$$= \frac{a_0}{2}(2\pi) = a_0\pi$$

$$\Longrightarrow \quad a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

Note: $\quad \dfrac{a_0}{2} = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,dx$ the average value of f on $(-\pi, \pi)$

Finding Fourier Coefficients

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

There's nothing special about b_4 or a_0 (aside from the factor of 2). In fact, for every $n = 1, 2, 3, \ldots$

$$\int_{-\pi}^{\pi} \cos^2(nx)\, dx = \pi, \quad \int_{-\pi}^{\pi} \sin^2(nx)\, dx = \pi$$

$$\text{and} \quad \int_{-\pi}^{\pi} 1^2\, dx = 2\pi.$$

The orthogonality property of the set $\{1, \cos(nx), \sin(nx) \,|\, n = 1, 2, \ldots\}$ provides a set of formulas for the coefficients, a_n, b_n.

The Fourier Series of $f(x)$ on $(-\pi, \pi)$

The **Fourier series** of the function f defined on $(-\pi, \pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right).$$

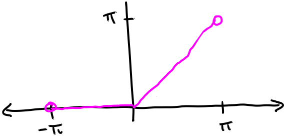
Where

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx, \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad \text{and} \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \end{aligned}$$

Example

Find the Fourier series of the piecewise defined function

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 \le x < \pi \end{cases}$$



Let's compute the a's + b's.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} 0\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\,dx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \frac{1}{\pi}\left[\frac{\pi^2}{2} - 0\right] = \frac{\pi}{2}$$

$$\boxed{a_0 = \frac{\pi}{2}}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} 0\cdot\cos(nx)\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\cos(nx)\,dx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} x\cos(nx)\,dx$$

Int by parts

$u = x$, $du = dx$

$dv = \cos(nx)\,dx$

$v = \frac{1}{n}\sin(nx)$

$$= \frac{1}{\pi}\left[\frac{x}{n}\sin(nx)\Big|_0^{\pi} - \int_0^{\pi}\frac{1}{n}\sin(nx)\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{x}{n}\sin(nx)\Big|_0^{\pi} + \frac{1}{n^2}\cos(nx)\Big|_0^{\pi}\right.$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}\sin(n\pi) - \frac{0}{n}\sin(0) + \frac{1}{n^2}\cos(n\pi) - \frac{1}{n^2}\cos(0)\right]$$

$$\sin(n\pi) = 0, \quad \cos(n\pi) = \begin{cases} -1, & n\text{-odd} \\ 1, & n\text{-even}\end{cases} = (-1)^n$$

$$a_n = \frac{1}{\pi}\left[\frac{1}{n^2}(-1)^n - \frac{1}{n^2}\right] = \frac{(-1)^n - 1}{n^2\pi}$$

$$\boxed{a_n = \frac{(-1)^n - 1}{n^2\pi}}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} 0\cdot\sin(nx)\,dx + \frac{1}{\pi}\int_0^{\pi} x\sin(nx)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\sin(nx)\,dx$$

by parts

$u = x \qquad du = dx$

$dv = \sin(nx)\,dx$

$v = \frac{-1}{n}\cos(nx)$

$$= \frac{1}{\pi}\left[\frac{-x}{n}\cos(nx)\Big|_0^{\pi} + \int_0^{\pi}\frac{1}{n}\cos(nx)\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{-x}{n}\cos(nx) + \frac{1}{n^2}\sin(nx)\right|_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-\pi}{n}\cos(n\pi) + \frac{1}{n^2}\sin(n\pi) - \frac{-0}{n}\cos(0) - \frac{1}{n^2}\sin(0)\right]$$

$$= \frac{1}{\pi}\left[\frac{-\pi}{n}(-1)^n\right] = \frac{-(-1)^n}{n} = \frac{(-1)^{n+1}}{n}$$

$$\boxed{b_n = \frac{(-1)^{n+1}}{n}}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx) + b_n\sin(nx)$$

$$a_0 = \frac{\pi}{2}, \quad a_n = \frac{(-1)^n - 1}{n^2 \pi}$$

$$\boxed{f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2 \pi} \cos(nx) + \frac{(-1)^{n+1}}{n} \sin(nx)}$$

This is the Fourier Series for

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 \le x < \pi \end{cases}$$

An Orthogonal Set of Functions on $[-p, p]$

This set can be generalized by using a simple change of variables $t = \frac{\pi x}{p}$ to obtain the orthogonal set on $[-p, p]$

$$\left\{1, \cos\frac{n\pi x}{p}, \sin\frac{m\pi x}{p} \,\middle|\, n, m = \pm 1, \pm 2, \dots\right\}$$

There are many interesting and useful orthogonal sets of functions (on appropriate intervals). What follows is readily extended to other such (infinite) sets.

Fourier Series on an interval $(-p, p)$

The set of functions $\left\{1, \cos\left(\frac{n\pi x}{p}\right), \sin\left(\frac{m\pi x}{p}\right) | n, m \geq 1\right\}$ is orthogonal on $[-p, p]$. Moreover, we have the properties

$$\int_{-p}^{p} \cos\left(\frac{n\pi x}{p}\right)\, dx = 0 \quad \text{and} \quad \int_{-p}^{p} \sin\left(\frac{m\pi x}{p}\right)\, dx = 0 \ \ \text{for all} \ \ n, m \geq 1,$$

$$\int_{-p}^{p} \cos\left(\frac{n\pi x}{p}\right) \sin\left(\frac{m\pi x}{p}\right)\, dx = 0 \quad \text{for all} \quad m, n \geq 1,$$

$$\int_{-p}^{p} \cos\left(\frac{n\pi x}{p}\right) \cos\left(\frac{m\pi x}{p}\right)\, dx = \begin{cases} 0, & m \neq n \\ p, & n = m \end{cases},$$

$$\int_{-p}^{p} \sin\left(\frac{n\pi x}{p}\right) \sin\left(\frac{m\pi x}{p}\right)\, dx = \begin{cases} 0, & m \neq n \\ p, & n = m \end{cases}.$$

Fourier Series on an interval $(-p, p)$

The orthogonality relations provide for an expansion of a function f defined on $(-p, p)$ as

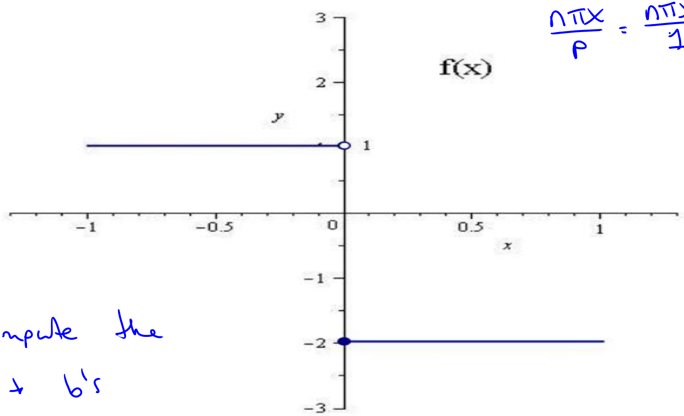
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{p}\right) + b_n \sin\left(\frac{n\pi x}{p}\right)\right)$$

where

$$\begin{aligned} a_0 &= \frac{1}{p}\int_{-p}^{p} f(x)\,dx, \\ a_n &= \frac{1}{p}\int_{-p}^{p} f(x)\cos\left(\frac{n\pi x}{p}\right)\,dx, \quad \text{and} \\ b_n &= \frac{1}{p}\int_{-p}^{p} f(x)\sin\left(\frac{n\pi x}{p}\right)\,dx \end{aligned}$$

Find the Fourier series of f

$$f(x) = \begin{cases} 1, & -1 < x < 0 \\ -2, & 0 \le x < 1 \end{cases}$$



f is defined on $(-p, p)$

$p = 1$

$\frac{n\pi x}{p} = \frac{n\pi x}{1} = n\pi x$

we'll compute the a's & b's

$$a_0 = \frac{1}{1}\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} 1\,dx + \int_{0}^{1} (-2)\,dx$$

$$= x\Big|_{-1}^{0} - 2x\Big|_{0}^{1} = (0-(-1)) - 2(1-0) = -1$$

$$\boxed{a_0 = -1}$$

$$a_n = \frac{1}{1}\int_{-1}^{1} f(x)\cos\left(\frac{n\pi x}{1}\right)dx$$

$$= \int_{-1}^{0} \cos(n\pi x)\,dx + \int_{0}^{1} (-2)\cos(n\pi x)\,dx$$

$$= \frac{1}{n\pi}\sin(n\pi x)\Big|_{-1}^{0} - \frac{2}{n\pi}\sin(n\pi x)\Big|_{0}^{1}$$

$$= \frac{1}{n\pi}\left(\sin 0 - \sin(-n\pi)\right) - \frac{2}{n\pi}\left(\sin(n\pi) - \sin(0)\right) = 0$$

$$\boxed{a_n = 0 \quad \text{for} \quad n = 1, 2, 3, \ldots}$$

$$b_n = \frac{1}{1}\int_{-1}^{1} f(x)\sin\left(\frac{n\pi x}{1}\right)dx$$

$$= \int_{-1}^{0} \sin(n\pi x)\,dx + \int_{0}^{1} (-2)\sin(n\pi x)\,dx$$

$$= \frac{-1}{n\pi}\cos(n\pi x)\Big|_{-1}^{0} + \frac{2}{n\pi}\cos(n\pi x)\Big|_{0}^{1}$$

$$= \frac{-1}{n\pi}\left[\cos 0 - \cos(-n\pi)\right] + \frac{2}{n\pi}\left[\cos(n\pi) - \cos 0\right]$$

$$= \frac{-1}{n\pi}\left(1-(-1)^n\right) + \frac{2}{n\pi}\left((-1)^n - 1\right)$$

$$= \frac{-1}{n\pi} + \frac{(-1)^n}{n\pi} + \frac{2(-1)^n}{n\pi} - \frac{2}{n\pi}$$

$$\boxed{b_n = \frac{3\left((-1)^n - 1\right)}{n\pi}}$$

$a_0 = -1$, $\quad a_n = 0 \quad n \geq 1$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) + b_n \sin(n\pi x)$$

Hence

$$\boxed{f(x) \doteq \frac{-1}{2} + \sum_{n=1}^{\infty} \frac{3\left((-1)^n-1\right)}{n\pi} \sin(n\pi x)}$$

$$f(x) = \begin{cases} 1, & -1 < x < 0 \\ -2, & 0 \le x < \underline{1} \end{cases}$$