November 12 Math 2306 sec. 54 Fall 2021

Section 17: Fourier Series: Trigonometric Series

Consider a function $f(x)$ defined on the interval $[-\pi, \pi]$. A series of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right).$$

is called the **Fourier series** or **Trigonometric series** of f.

An Orthogonal Set of Functions on $[-\pi, \pi]$

Recall that the set of functions

$$\{1, \cos x, \cos 2x, \cos 3x, \ldots, \sin x, \sin 2x, \sin 3x, \ldots\}$$

is an orthogonal set on the interval $[-\pi, \pi]$.

Key Point: This means that if we take any two functions f and g **from this set**, then

$$\int_{-\pi}^{\pi} f(x)g(x)\,dx = 0 \quad \text{if } f \text{ and } g \text{ are different functions!}$$

Fourier Series

Suppose $f(x)$ is defined for $-\pi < x < \pi$. We would like to know how to write f as a series **in terms of sines and cosines.**

Task: Find coefficients (numbers) a_0, $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ such that[1]

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right).$$

[1] We'll write $\frac{a_0}{2}$ as opposed to a_0 purely for convenience.

Finding an Example Coefficient

Let's find the coefficient b_4.

Start with the series $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$, and multiply both sides by $\sin(4x)$.

$$f(x)\sin(4x) = \frac{a_0}{2}\sin(4x) + \sum_{n=1}^{\infty}\left(a_n \cos nx \sin(4x) + b_n \sin nx \sin(4x)\right).$$

We integrated from $-\pi$ to π. Using the orthogonality property, we came to the conclusion that

$$\int_{-\pi}^{\pi} f(x)\sin(4x)\,dx = \pi b_4$$

$$\frac{\langle f(x), \sin(4x)\rangle}{\langle \sin(4x), \sin(4x)\rangle}$$

giving us a formula

$$b_4 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(4x)\,dx.$$

Finding an Example Coefficient

Let's find the constant term $a_0/2$. We'll follow the same procedure. Start by assuming f has the series representation.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right).$$

- Multiply both sides of the equation by 1,
- integrate from $-\pi$ to π.

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} \frac{a_0}{2}\,dx + \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} a_n \cos(nx)\,dx + \int_{-\pi}^{\pi} b_n \sin(nx)\,dx$$

By orthogonality

$$\int_{-\pi}^{\pi} \cos(nx)\,dx = 0 \quad \text{and} \quad \int_{-\pi}^{\pi} \sin(nx)\,dx = 0 \quad \text{for all } n=1,2,3,\dots$$

This reduces to

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} \frac{a_0}{2}\,dx = \frac{a_0}{2}\int_{-\pi}^{\pi} dx$$

$$= \frac{a_0}{2}\left[x\Big|_{-\pi}^{\pi}\right] = \frac{a_0}{2}\left[\pi - (-\pi)\right]$$

$$= \frac{a_0}{2}(2\pi) = a_0\pi$$

$$\implies a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

Note: $\frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$ is the average value of f on $(-\pi, \pi)$

Finding Fourier Coefficients

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

There's nothing special about b_4 or a_0 (aside from the factor of 2). In fact, for every $n = 1, 2, 3, \ldots$

$$\int_{-\pi}^{\pi} \cos^2(nx)\, dx = \pi, \quad \int_{-\pi}^{\pi} \sin^2(nx)\, dx = \pi$$

$$\text{and} \quad \int_{-\pi}^{\pi} 1^2\, dx = 2\pi.$$

The orthogonality property of the set $\{1, \cos(nx), \sin(nx) \mid n = 1, 2, \ldots\}$ provides a set of formulas for the coefficients, a_n, b_n.

The Fourier Series of $f(x)$ on $(-\pi, \pi)$

The **Fourier series** of the function f defined on $(-\pi, \pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right).$$

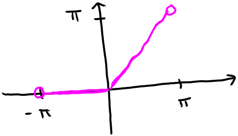
Where

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx, \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad \text{and} \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \end{aligned}$$

Example

Find the Fourier series of the piecewise defined function

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 \le x < \pi \end{cases}$$



Let's find the a's and b's.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} 0\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2}\Big|_0^{\pi}\right] = \frac{1}{\pi}\left[\frac{\pi^2}{2} - 0\right] = \frac{\pi}{2}$$

$$\boxed{a_0 = \frac{\pi}{2}}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} 0\cdot\cos(nx)\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\cos(nx)\,dx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} x\cos(nx)\,dx$$

Int by parts

$u = x, \quad du = dx$

$dv = \cos(nx)\,dx$

$v = \frac{1}{n}\sin(nx)$

$$= \frac{1}{\pi}\left[\frac{x}{n}\sin(nx)\Big|_{0}^{\pi} - \int_{0}^{\pi}\frac{1}{n}\sin(nx)\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{x}{n}\sin(nx) + \frac{1}{n^2}\cos(nx)\Big|_{0}^{\pi}\right.$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}\sin(n\pi) + \frac{1}{n^2}\cos(n\pi) - \frac{0}{n}\sin(0) - \frac{1}{n^2}\cos(0)\right]$$

Recall $\sin(n\pi) = 0$ for all $n = \pm 1, \pm 2, \ldots$

$$\cos(n\pi) = \begin{cases} -1, & n\text{-odd} \\ 1, & n\text{-even} \end{cases} = (-1)^n$$

$$a_n = \frac{1}{\pi}\left[0 + \frac{1}{n^2}(-1)^n - 0 - \frac{1}{n^2}\right] = \frac{(-1)^n - 1}{n^2 \pi}$$

$$\boxed{a_n = \frac{(-1)^n - 1}{n^2 \pi}}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} 0\cdot\sin(nx)\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\sin(nx)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\sin(nx)\,dx$$

Int by parts

$u = x \quad du = dx$

$dv = \sin(nx)\,dx$

$v = \frac{-1}{n}\cos(nx)$

$$= \frac{1}{\pi}\left[\frac{-x}{n}\cos(nx)\Big|_0^{\pi} + \int_0^{\pi}\frac{1}{n}\cos(nx)\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{-x}{n}\cos(nx) + \frac{1}{n^2}\sin(nx)\right|_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-\pi}{n}\cos(n\pi) + \frac{1}{n^2}\sin(n\pi) - \frac{-0}{n}\cos(0) - \frac{1}{n^2}\sin(0)\right]$$

$$= \frac{1}{\pi}\left[\frac{-\pi}{n}(-1)^n\right] = \frac{-(-1)^n}{n} = \frac{(-1)^{n+1}}{n}$$

$$\boxed{b_n = \frac{(-1)^{n+1}}{n}} \qquad \boxed{a_0 = \frac{\pi}{2}} \qquad \boxed{a_n = \frac{(-1)^n - 1}{n^2\pi}}$$

The seies has the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

Hence

$$\boxed{f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2 \pi} \cos(nx) + \frac{(-1)^{n+1}}{n} \sin(nx)}$$

where

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 \leq x < \pi \end{cases}$$

An Orthogonal Set of Functions on $[-p, p]$

This set can be generalized by using a simple change of variables $t = \frac{\pi x}{p}$ to obtain the orthogonal set on $[-p, p]$

$$\left\{1, \cos\frac{n\pi x}{p}, \sin\frac{m\pi x}{p} \,\middle|\, n, m = \pm 1, \pm 2, \dots\right\}$$

There are many interesting and useful orthogonal sets of functions (on appropriate intervals). What follows is readily extended to other such (infinite) sets.

Fourier Series on an interval $(-p, p)$

The set of functions $\left\{1, \cos\left(\frac{n\pi x}{p}\right), \sin\left(\frac{m\pi x}{p}\right) | n, m \geq 1\right\}$ is orthogonal on $[-p, p]$. Moreover, we have the properties

$$\int_{-p}^{p} \cos\left(\frac{n\pi x}{p}\right)\, dx = 0 \quad \text{and} \quad \int_{-p}^{p} \sin\left(\frac{m\pi x}{p}\right)\, dx = 0 \;\; \text{for all} \;\; n, m \geq 1,$$

$$\int_{-p}^{p} \cos\left(\frac{n\pi x}{p}\right) \sin\left(\frac{m\pi x}{p}\right)\, dx = 0 \quad \text{for all} \quad m, n \geq 1,$$

$$\int_{-p}^{p} \cos\left(\frac{n\pi x}{p}\right) \cos\left(\frac{m\pi x}{p}\right)\, dx = \begin{cases} 0, & m \neq n \\ p, & n = m \end{cases},$$

$$\int_{-p}^{p} \sin\left(\frac{n\pi x}{p}\right) \sin\left(\frac{m\pi x}{p}\right)\, dx = \begin{cases} 0, & m \neq n \\ p, & n = m \end{cases}.$$

Fourier Series on an interval $(-p, p)$

The orthogonality relations provide for an expansion of a function f defined on $(-p, p)$ as

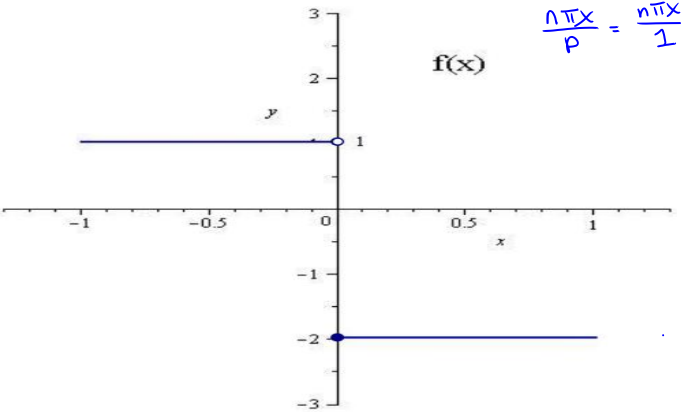
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{p}\right) + b_n \sin\left(\frac{n\pi x}{p}\right)\right)$$

where

$$\begin{aligned} a_0 &= \frac{1}{p}\int_{-p}^{p} f(x)\,dx, \\ a_n &= \frac{1}{p}\int_{-p}^{p} f(x)\cos\left(\frac{n\pi x}{p}\right)\,dx, \quad \text{and} \\ b_n &= \frac{1}{p}\int_{-p}^{p} f(x)\sin\left(\frac{n\pi x}{p}\right)\,dx \end{aligned}$$

Find the Fourier series of f

$$f(x) = \begin{cases} 1, & -1 < x < 0 \\ -2, & 0 \le x < 1 \end{cases}$$



f is defined on $(-p, p)$

$p = 1$

$$\frac{n\pi x}{p} = \frac{n\pi x}{1} = n\pi x$$