November 13 Math 2306 sec. 52 Spring 2023

Section 15: Shift Theorems

We saw the first translation theorem.

Theorem: Shift in s

Suppose $\mathscr{L}\{f(t)\} = F(s)$. Then for any real number a

$$\mathscr{L}\left\{e^{at}f(t)\right\} = F(s-a).$$

Equivalently,

$$\mathscr{L}^{-1}\{F(s-a)\} = e^{at}\mathscr{L}^{-1}\{F(s)\}.$$

We'll see a similar result relating to translations in the variable t. This requires some preliminary ideas.

The Unit Step Function

Definition: Unit Step Function

Let $a > 0$. The unit step function *centered at a* is denoted $\mathscr{U}(t-a)$. It is defined by

$$\mathscr{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a \end{cases}$$

The name *unit step function* is derived from its graph which looks like a stair step of height 1.

$\mathcal{U}(t) = \mathcal{U}(t-0)$

$= 1$ for all $t \geq 0$

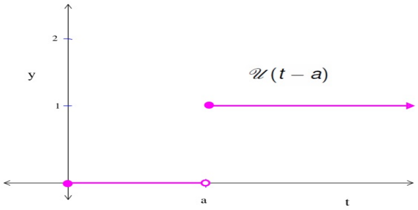

Figure: A graph of $\mathscr{U}(t-a)$ which jumps from zero to one at $t = a$.

Piecewise Defined Functions

Verify that

$$f(t) = \begin{cases} g(t), & 0 \le t < a \\ h(t), & t \ge a \end{cases} \qquad a > 0$$

$$= g(t) - g(t)\mathscr{U}(t-a) + h(t)\mathscr{U}(t-a)$$

Suppose t is in the interval $0 \le t < a$

Then $\mathscr{U}(t-a) = 0$

$g(t) - g(t)\mathscr{U}(t-a) + h(t)\mathscr{U}(t-a) =$

$g(t) - g(t) \cdot 0 + h(t) \cdot 0 = g(t)$

This is equal to $f(t)$ on this interval.

$$\begin{cases} g(t), & 0 \leq t < a \\ h(t), & t \geq a \end{cases} = g(t) - g(t)\mathscr{U}(t-a) + h(t)\mathscr{U}(t-a)$$

Suppose $t \geq a$, then $\mathcal{U}(t-a) = 1$

And

$g(t) - g(t)\mathcal{U}(t-a) + h(t)\mathcal{U}(t-a) =$

$g(t) - g(t) \cdot 1 + h(t) \cdot 1 \quad = h(t)$

This is also equal to $f(t)$

on this interval

$$\text{Example } f(t) = \begin{cases} e^t, & 0 \le t < 2 \\ t^2, & 2 \le t < 5 \\ 2t & t \ge 5 \end{cases}$$

Rewrite the function f in terms of the unit step function.

$f(t) = e^t - e^t u(t-2) + t^2 u(t-2) - t^2 u(t-5) + 2t\, u(t-5)$

Check: Suppose $0 \le t < 2$, $u(t-2) = 0$ $\quad u(t-5) = 0$

$f(t) = e^t - e^t \cdot 0 + t^2 \cdot 0 - t^2 \cdot 0 + 2t \cdot 0 = e^t$

For $2 \le t < 5$, $u(t-2) = 1$ $\quad u(t-5) = 0$

$f(t) = e^t - e^t \cdot 1 + t^2 \cdot 1 - t^2 \cdot 0 + 2t \cdot 0 = t^2$

For $t \geq 5$, $u(t-2) = 1$ $\quad u(t-5) = 1$

$f(t) = e^t - e^t \cdot 1 + t^2 \cdot 1 - t^2 \cdot 1 + 2t \cdot 1 = 2t$

Find $\mathscr{L}\{\mathscr{U}(t-a)\}$

$a > 0$

$$\mathcal{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a \end{cases}$$

By definition

$$\mathcal{L}\{\mathcal{U}(t-a)\} = \int_0^\infty e^{-st}\,\mathcal{U}(t-a)\,dt$$

$$= \underbrace{\int_0^a e^{-st}\cdot 0\,dt}_{0} + \int_a^\infty e^{-st}\cdot 1\,dt$$

for $s > 0$

$$= \frac{-1}{s}\,e^{-st}\Big|_a^\infty = \frac{-1}{s}\left(0 - e^{-s(a)}\right)$$

$$= \frac{e^{-as}}{s}$$

Translation in t

Given a function $f(t)$ for $t \geq 0$, and a number $a > 0$

$$f(t-a)\mathscr{U}(t-a) = \begin{cases} 0, & 0 \leq t < a \\ f(t-a), & t \geq a \end{cases}.$$

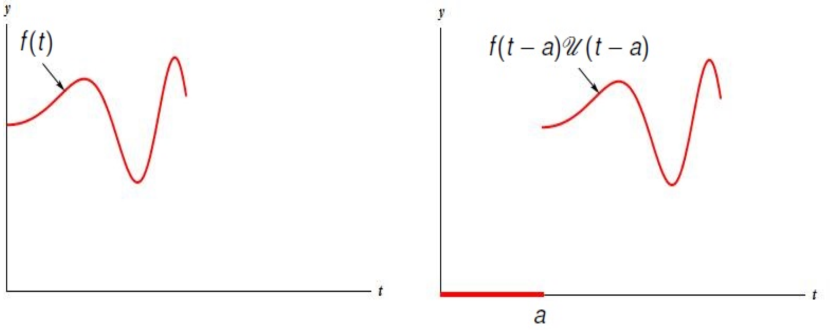

Figure: The function $f(t-a)\mathscr{U}(t-a)$ has the graph of f shifted a units to the right with value of zero for t to the left of a.

Theorem (translation in t)

If $F(s) = \mathscr{L}\{f(t)\}$ and $a > 0$, then

$$\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} = e^{-as}F(s).$$

A special case is $f(t) = 1$. We just found
$\mathscr{L}\{\mathscr{U}(t-a)\} = e^{-as}\mathscr{L}\{1\} = \dfrac{e^{-as}}{s}$.

We can state this in terms of the inverse transform as

$$\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathscr{U}(t-a).$$