November 13 Math 2306 sec. 53 Fall 2024

Section 15: Shift Theorems

Theorem Shift in s:

Suppose $\mathscr{L}\{f(t)\} = F(s)$. Then for any real number a

$$\mathscr{L}\left\{e^{at}f(t)\right\} = F(s-a).$$

From the perspective of the inverse transform

$$\mathscr{L}^{-1}\{F(s-a)\} = e^{at}\mathscr{L}^{-1}\{F(s)\}.$$

The Unit Step Function, Piecewise Defined Functions & t-Shift

We defined the unit step function $\mathscr{U}(t-a)$ as

$$\mathscr{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a \end{cases}$$

Piecewise defined functions can be expressed in terms of unit step factors.

Given a function $f(t)$ defined for $t \ge 0$ and $a > 0$, we defined the translation

$$f(t-a)\mathscr{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ f(t-a), & t \ge a \end{cases}.$$

which translates f to the right a units and assigns the value of zero on the interval $[0, a)$.

Theorem (translation in t)

Then we have the theorem

Theorem Shift in t:

If $F(s) = \mathscr{L}\{f(t)\}$ and $a > 0$, then

$$\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} = e^{-as}F(s).$$

We can state this in terms of the inverse transform as

$$\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathscr{U}(t-a).$$

Alternative Form for Translation in t

It is often the case that we wish to take the transform of a product of the form

$$g(t)\mathscr{U}(t-a)$$

in which the function g is not translated.

The main theorem statement

$$\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} = e^{-as}F(s).$$

can be restated as

$$\boxed{\mathscr{L}\{g(t)\mathscr{U}(t-a)\} = e^{-as}\mathscr{L}\{g(t+a)\}.}$$

This is based on the observation that

$$g(t) = g((t+a)-a).$$

Example

$$\mathscr{L}\{g(t)\mathscr{U}(t-a)\} = e^{-as}\mathscr{L}\{g(t+a)\}$$

$a = \frac{\pi}{2}$, $g(t) = \cos t$

Example: Find $\mathscr{L}\{\cos t\, \mathscr{U}\left(t - \frac{\pi}{2}\right)\}$

$= e^{-\frac{\pi}{2}s} \mathcal{L}\{\cos(t + \frac{\pi}{2})\}$

$= e^{-\frac{\pi}{2}s} \mathcal{L}\{-\sin t\}$

$= -e^{-\frac{\pi}{2}s} \mathcal{L}\{\sin t\} = -e^{-\frac{\pi}{2}s}\left(\frac{1}{s^2+1^2}\right)$

$= \dfrac{-e^{-\frac{\pi}{2}s}}{s^2+1}$

$$\cos\left(t+\frac{\pi}{2}\right)=\cos t\,\underbrace{\cos\frac{\pi}{2}}_{0}-\sin t\,\underbrace{\sin\frac{\pi}{2}}_{1}=-\sin t$$

Example

$$\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathscr{U}(t-a)$$

Example: Find $\mathscr{L}^{-1}\left\{\dfrac{e^{-2s}}{s(s+1)}\right\}$ $= \mathscr{L}^{-1}\left\{ e^{-2s} \dfrac{1}{s(s+1)} \right\}$

Find $\hat{f}(t) = \mathscr{L}^{-1}\left\{\dfrac{1}{s(s+1)}\right\} = \mathscr{L}^{-1}\left\{\dfrac{1}{s}\left(\dfrac{1}{s+1}\right)\right\}$

$\mathscr{L}^{-1}\{F(s)G(s)\} = (f * g)(t)$

$F(s) = \dfrac{1}{s+1} \Rightarrow f(t) = \mathscr{L}^{-1}\left\{\dfrac{1}{s+1}\right\} = e^{-t}$

$G(s) = \dfrac{1}{s} \Rightarrow g(t) = \mathscr{L}^{-1}\left\{\dfrac{1}{s}\right\} = 1$

$$(f*g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau = \int_0^t e^{-\tau}\,1\,d\tau$$

$$= -e^{-\tau}\Big|_0^t = -e^{-t} - (-e^{0}) = 1 - e^{-t}$$

$$\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathscr{U}(t-a)$$

$$\mathscr{L}^{-1}\left\{\frac{e^{-2s}}{s(s+1)}\right\} = f(t-2)\,\mathscr{U}(t-2)$$

$$= \left(1 - e^{-(t-2)}\right)\mathscr{U}(t-2)$$

Example

$$\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathscr{U}(t-a)$$

Example: Evaluate $\mathscr{L}^{-1}\left\{\dfrac{e^{-4s}}{(s+6)^9}\right\}$

we need $\mathscr{L}^{-1}\left\{\dfrac{1}{(s+6)^9}\right\} = f(t)$

$\dfrac{1}{(s+6)^9} \to \dfrac{1}{s^9}$ with $s-(-6)$ in place of s.

$$\mathscr{L}^{-1}\left\{\frac{1}{(s+6)^9}\right\} = e^{-6t}\mathscr{L}^{-1}\left\{\frac{1}{s^9}\right\}$$

$$= e^{-6t}\mathscr{L}^{-1}\left(\frac{1}{8!}\,\frac{8!}{s^9}\right)$$

$$= \frac{1}{8!} e^{-6t} \mathcal{L}^{-1}\left\{ \frac{8!}{s^9} \right\} = \frac{1}{8!} e^{-6t} t^8$$

$$f(t) = \frac{1}{8!} e^{-6t} t^8$$

$$\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathscr{U}(t-a)$$

$$\mathscr{L}^{-1}\left\{\frac{e^{-4s}}{(s+6)^9}\right\} = \frac{1}{8!} e^{-6(t-4)} (t-4)^8 \mathscr{U}(t-4)$$

Section 16: Laplace Transforms of Derivatives and IVPs

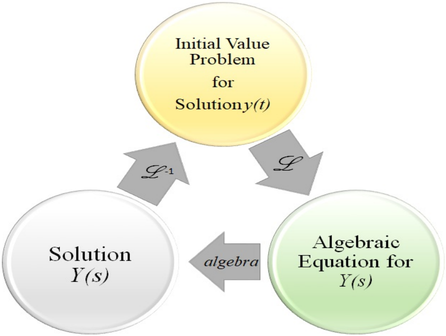

Figure: We'll use the Laplace transform as a tool for solving certain IVPs and systems of IVPs. Our use will be restricted to IVPs with **constant coefficients** and initial conditions given at $t = 0$.

First: Let's look at differentiation.

Transforms of Derivatives

We saw[1] how the following is obtained from the definition of the Laplace transform and a bit of integration by parts.

The Laplace Transform of a Derivative

Suppose f is differentiable on $[0, \infty)$ and $F(s) = \mathscr{L}\{f(t)\}$, then

$$\mathscr{L}\{f'(t)\} = sF(s) - f(0).$$

We can use this result recursively to get transforms for higher order derivatives.

[1] See Worksheet 14 for details.

Transforms of Derivatives

Suppose $F(s) = \mathscr{L}\{f(t)\}$ so that $\mathscr{L}\{f'(t)\} = sF(s) - f(0)$. Express $\mathscr{L}\{f''(t)\}$ in terms of F.

$$\mathscr{L}\{f''(t)\} = s\,\mathscr{L}\{f'(t)\} - f'(0)$$

$$= s\left(sF(s) - f(0)\right) - f'(0)$$

$$= s^2 F(s) - sf(0) - f'(0)$$

Remark

Note that the operation of differentiation where the variable t lives corresponds to an algebraic operation, *multiply by some power of s and add a polynomial*, where s lives.

The Laplace Transform of Derivatives

For $y = y(t)$ defined on $[0, \infty)$ having derivatives y', y'' and so forth, if $\mathscr{L}\{y(t)\} = Y(s)$, then

$$\begin{aligned}
\mathscr{L}\left\{\frac{dy}{dt}\right\} &= sY(s) - y(0) \\
\mathscr{L}\left\{\frac{d^2y}{dt^2}\right\} &= s^2Y(s) - sy(0) - y'(0) \\
\mathscr{L}\left\{\frac{d^3y}{dt^3}\right\} &= s^3Y(s) - s^2y(0) - sy'(0) - y''(0) \\
&\vdots \\
\mathscr{L}\left\{\frac{d^ny}{dt^n}\right\} &= s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - y^{(n-1)}(0).
\end{aligned}$$

Warning about Notation

- The characters Y and y DO NOT represent the same thing. They CANNOT be used interchangeably.

- An expression such as $y'(0)$ means the value of the function $y'(t)$ when the input $t = 0$.

- The function $\mathscr{L}\{y(t)\}$ depends on s NOT on t.

- And, the function $\mathscr{L}^{-1}\{Y(s)\}$ depends on t NOT on s.

Solving and IVP

Use the Laplace transform to solve the initial value problem.

$$y'' - 6y' + 8y = t, \quad y(0) = 1, \quad y'(0) = 2$$

Take transform of both sides of the ODE.

Let $Y(s) = \mathcal{L}\{y(t)\}$.

$$\mathcal{L}\{y'' - 6y' + 8y\} = \mathcal{L}\{t\}$$

$$\mathcal{L}\{y''\} - 6\mathcal{L}\{y'\} + 8\mathcal{L}\{y\} = \mathcal{L}\{t\}$$

$$s^2Y(s) - s\underbrace{y(0)}_{1} - \underbrace{y'(0)}_{2} - 6\left(sY(s) - \underbrace{y(0)}_{1}\right) + 8Y(s) = \frac{1!}{s^2}.$$

$$s^2 Y(s) - s - 2 - 6s\,Y(s) + 6 + 8\,Y(s) = \frac{1}{s^2}$$

Isolate $Y(s)$.

$$s^2 Y(s) - 6s\,Y(s) + 8\,Y(s) - s + 4 = \frac{1}{s^2}$$

$$\underbrace{(s^2 - 6s + 8)}\,Y(s) = s - 4 + \frac{1}{s^2}$$

$$y'' - 6y' + 8y = t,$$

this is the characteristic polynomial for the ODE.

$$\boxed{Y(s) = \frac{s-4}{s^2 - 6s + 8} + \frac{\frac{1}{s^2}}{s^2 - 6s + 8}}$$

Y is $\mathcal{L}\{y(t)\}$ so we want $y(t) = \mathcal{L}^{-1}\{Y(s)\}$.

$$Y(s) = \frac{s-4}{(s-2)(s-4)} + \frac{1}{s^2(s-2)(s-4)}$$

Using partial fractions

$$\frac{1}{s^2(s-2)(s-4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-2} + \frac{D}{s-4}$$

after some work $A = \frac{3}{32}$, $B = \frac{1}{8}$, $C = \frac{-1}{8}$, $D = \frac{1}{32}$

$$Y(s) = \frac{1}{s-2} + \frac{\frac{3}{32}}{s} + \frac{\frac{1}{8}}{s^2} - \frac{\frac{1}{8}}{s-2} + \frac{\frac{1}{32}}{s-4}$$

$$Y(s) = \frac{\frac{7}{8}}{s-2} + \frac{\frac{3}{32}}{s} + \frac{\frac{1}{8}}{s^2} + \frac{\frac{1}{32}}{s-4}$$

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{\frac{7}{8}}{s-2} + \frac{\frac{3}{32}}{s} + \frac{\frac{1}{8}}{s^2} + \frac{\frac{1}{32}}{s-4} \right\}$$

$$= \frac{7}{8}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{3}{32}\mathcal{L}^{-1}\left(\frac{1}{s}\right) + \frac{1}{8}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \frac{1}{32}\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\}$$

$$\boxed{y(t) = \frac{7}{8}e^{2t} + \frac{3}{32} + \frac{1}{8}t + \frac{1}{32}e^{4t}}$$

Check: $y(0) = 1$?

$$y(0) = \frac{7}{8}e^0 + \frac{3}{32} + \frac{1}{8}(0) + \frac{1}{32}e^0$$

$$= \frac{7}{8} + \frac{3}{32} + \frac{1}{32} = \frac{7}{8} + \frac{4}{32} = \frac{7}{8} + \frac{1}{8} = \frac{8}{8} = 1$$

Use Laplace Transforms to Solve and IVP

- Start with constant coefficient IVP with IC at $t = 0$. For example[a]

$$ay'' + by' + cy = g(t), \quad y(0) = y_0, \quad y'(0) = y_1.$$

- Let $Y(s) = \mathscr{L}\{y(t)\}$ and take the transform of both sides of the ODE using any necessary results.
- Sub in the initial conditions where they appear in the transformed equation.
- Use basic algebra to isolate the transform $Y(s)$.
- Using whatever algebra or function identities that are needed, take the inverse transform to obtain the solution

$$y(t) = \mathscr{L}^{-1}\{Y(s)\}.$$

[a] The IVP can be of any order.

Input & State Responses

Note that our solution has the basic format

$$Y(s) = \frac{Q(s)}{P(s)} + \frac{G(s)}{P(s)}$$

where Q is a polynomial with coefficients determined by the initial conditions, G is the Laplace transform of $g(t)$ and P is the **characteristic polynomial** of the original equation. The solution y consists of two corresponding terms.

Zero Input Response

$$\mathscr{L}^{-1}\left\{\frac{Q(s)}{P(s)}\right\}$$ is called the **zero input response**,

and

Zero State Response

$$\mathscr{L}^{-1}\left\{\frac{G(s)}{P(s)}\right\}$$ is called the **zero state response**.

Input & State Responses

The zero input and zero state responses are related to the complementary and particular solutions, but they are not quite the same since they are related to initial value problems as opposed to simply the differential equation.

Zero Input Response

The **zero input response** satisfies the initial value problem with homogeneous differential equation and nonhomogeneous initial conditions,

$$ay'' + by' + cy = 0, \quad y(0) = y_0, \quad y'(0) = y_1.$$

Zero State Response

The **zero state response** satisfies the initial value problem with nonhomogeneous differential equation and homogeneous initial conditions,

$$ay'' + by' + cy = g(t), \quad y(0) = 0, \quad y'(0) = 0.$$