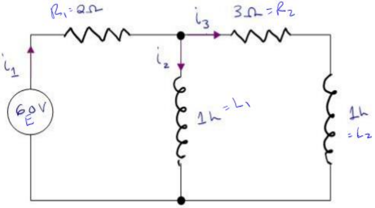
November 14 Math 2306 sec. 52 Fall 2022

Section 16: Laplace Transforms of Derivatives and IVPs

We can solve a system of ODEs using Laplace transforms. Here, we'll consider systems that are

- linear,
- having initial conditions at $t = 0$, and
- constant coefficient.

Example



$$
\begin{aligned}
L_1 i_2' &+ R_1(i_2+i_3) & &= E \\
L_2 i_3' &+ R_1(i_2+i_3) &+ R_2 i_3 &= E
\end{aligned}
$$

Figure: If we label current i_2 as $x(t)$ and current i_3 as $y(t)$, we get the system of equations below. (Assuming $i_1(0) = 0$.)

Solve the system of equations

$$
\begin{aligned}
\frac{dx}{dt} &= -2x - 2y + 60, \quad x(0) = 0 \\
\frac{dy}{dt} &= -2x - 5y + 60, \quad y(0) = 0
\end{aligned}
$$

$$\begin{aligned} \frac{dx}{dt} &= -2x - 2y + 60, \quad x(0) = 0 \\ \frac{dy}{dt} &= -2x - 5y + 60, \quad y(0) = 0 \end{aligned}$$

We took the Laplace transform of both equations and used the initial conditions. We arrived at the following linear, algebraic system of equations.

$$\begin{aligned} (s+2)X(s) &+ \quad 2Y(s) &= \frac{60}{s} \\ 2X(s) &+ (s+5)Y(s) &= \frac{60}{s} \end{aligned} \quad \text{where} \quad \begin{aligned} X(s) &= \mathscr{L}\{x(t)\} \\ &\text{and} \\ Y(s) &= \mathscr{L}\{y(t)\} \end{aligned}$$

We can restate this using a matrix formalism.

Example Continued...

$$\begin{aligned}(s+2)X(s) &+ & 2Y(s) &= \frac{60}{s}\\ 2X(s) &+ & (s+5)Y(s) &= \frac{60}{s}\end{aligned}$$

$$\begin{bmatrix} s+2 & 2 \\ 2 & s+5 \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 60/s \\ 60/s \end{bmatrix}$$

$$A = \begin{bmatrix} s+2 & 2 \\ 2 & s+5 \end{bmatrix} \qquad A_y = \begin{bmatrix} s+2 & 60/s \\ 2 & 60/s \end{bmatrix}$$

$$A_x = \begin{bmatrix} 60/s & 2 \\ 60/s & s+5 \end{bmatrix}$$

$$\det(A) = (s+2)(s+5) - 4 = s^2 + 7s + 10 - 4 = s^2 + 7s + 6$$

$$\det(A_x) = \frac{60}{s}(s+5) - \frac{60}{s}(2) = \frac{60}{s}(s+5-2) = \frac{60}{s}(s+3)$$

$$\det(A_Y) = (s+2)\frac{60}{s} - 2\left(\frac{60}{s}\right) = \frac{60}{s}(s+2-2) = \frac{60}{s}(s) = 60$$

$$X(s) = \frac{\frac{60}{s}(s+3)}{s^2+7s+6} = \frac{60(s+3)}{s(s+1)(s+6)}$$

$$Y(s) = \frac{60}{s^2+7s+6} = \frac{60}{(s+1)(s+6)}$$

Using partial fraction decomps

$$X(s) = \frac{30}{s} - \frac{24}{s+1} - \frac{6}{s+6}$$

$$Y(s) = \frac{12}{s+1} - \frac{12}{s+6}$$

The solution to the IVP

$x(t) = \mathcal{L}^{-1}\{X(s)\}$ and $y(t) = \mathcal{L}^{-1}\{Y(s)\}$.

$$\boxed{\begin{aligned} x(t) &= 30 - 24e^{-t} - 6e^{-6t} \\ y(t) &= 12e^{-t} - 12e^{-6t} \end{aligned}}$$

$$\begin{aligned} \frac{dx}{dt} &= -2x - 2y + 60, \quad x(0) = 0 \\ \frac{dy}{dt} &= -2x - 5y + 60, \quad y(0) = 0 \end{aligned}$$

Example

Use the Laplace transform to solve the system of equations

$$\begin{aligned} x''(t) &= y, \quad x(0) = 1, \quad x'(0) = 0 \\ y'(t) &= x, \quad y(0) = 1 \end{aligned}$$

Let $X(s) = \mathcal{L}\{x(t)\}$ and $Y(s) = \mathcal{L}\{y(t)\}$.

$$\begin{aligned} \mathcal{L}\{x''\} &= \mathcal{L}\{y\} \\ \mathcal{L}\{y'\} &= \mathcal{L}\{x\} \end{aligned} \implies \begin{aligned} s^2X(s) - s\underbrace{x(0)}_{1} - \underbrace{x'(0)}_{0} &= Y(s) \\ sY(s) - \underbrace{y(0)}_{1} &= X(s) \end{aligned}$$

$$s^2X - s = Y$$

$$sY - 1 = X$$

$$s^2 X - Y = s$$
$$-X + sY = 1$$
$$\Longrightarrow \begin{bmatrix} s^2 & -1 \\ -1 & s \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} s \\ 1 \end{bmatrix}$$

Using Cramer's rule, let

$$A = \begin{bmatrix} s^2 & -1 \\ -1 & s \end{bmatrix} \qquad \det(A) = s^3 - 1$$

$$A_X = \begin{bmatrix} s & -1 \\ 1 & s \end{bmatrix} \qquad \det(A_X) = s^2 + 1$$

$$A_Y = \begin{bmatrix} s^2 & s \\ -1 & 1 \end{bmatrix} \qquad \det(A_Y) = s^2 + s$$

$$X(s) = \frac{s^2+1}{s^3-1} = \frac{s^2+1}{(s-1)(s^2+s+1)}$$

$$Y(s) = \frac{s^2+s}{s^3-1} = \frac{s^2+s}{(s-1)(s^2+s+1)}$$

$$\frac{s^2+1}{(s-1)(s^2+s+1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+s+1}$$

$$X(s) = \frac{\frac{2}{3}}{s-1} + \frac{\frac{1}{3}s - \frac{1}{3}}{s^2+s+1}$$

$$Y(s) = \frac{\frac{2}{3}}{s-1} + \frac{\frac{1}{3}s + \frac{2}{3}}{s^2+s+1}$$

$$s^2+s+1 = \left(s+\tfrac{1}{2}\right)^2 + \frac{3}{4} = \left(s+\tfrac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2$$

$$X(s) = \frac{\frac{2}{3}}{s-1} + \frac{1}{3}\,\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} - \frac{1}{\sqrt{3}}\,\frac{\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$Y(s) = \frac{\frac{2}{3}}{s-1} + \frac{1}{3}\,\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} + \frac{1}{\sqrt{3}}\,\frac{\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$\mathcal{L}^{-1}\left\{\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\} = e^{-\frac{1}{2}t}\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\}$$

The solution to the IVP

$$x(t) = \frac{2}{3}e^{t} + \frac{1}{3}e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{1}{\sqrt{3}}e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{2}t\right)$$

$$y(t) = \frac{2}{3}e^{t} + \frac{1}{3}e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{1}{\sqrt{3}}e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{2}t\right)$$

$$\begin{aligned} x''(t) &= y, \quad x(0) = 1, \quad x'(0) = 0 \\ y'(t) &= x, \quad y(0) = 1 \end{aligned}$$