November 14 Math 3260 sec. 53 Fall 2025

5.2 Linear Transformations for R^n to R^m

Linear Transformation

A **linear transformation** from R^n to R^m is a function $T : R^n \to R^m$ such that for each pair of vectors $\vec{x}$ and $\vec{y}$ in R^n and for any scalar c

1. $T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y})$, and
2. $T(c\vec{x}) = cT(\vec{x})$.

Lemma

If A is an $m \times n$ matrix and $T : R^n \to R^m$ is defined by $T(\vec{x}) = A\vec{x}$, then T is a linear transformation.

Linear Transformations & Matrices

Theorem

Suppose that $T : R^n \to R^m$ is a linear transformation. Then there is a unique $m \times n$ matrix A, such that $T(\vec{x}) = A\vec{x}$ for all $\vec{x} \in R^n$.

Furthermore, the matrix[a] A is the matrix whose column vectors are

$$\mathrm{Col}_j(A) = T(\vec{e}_j)$$

where $\mathcal{E} = \{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ is the standard basis for R^n.

[a] We'll call A the **standard matrix** for the transformation T.

Fundamental Subspaces: Range and Kernel

Let $T : R^n \to R^m$ be a linear transformation, and let A be its standard matrix. The **range** of T is defined by

$$\text{range}(T) = \{T(\vec{x}) \mid \vec{x} \in R^n\}.$$

and the **kernel** of T, denoted $\ker(T)$ is defined by

$$\ker(T) = \left\{\vec{x} \in R^n \mid T(\vec{x}) = \vec{0}_m\right\}.$$

Moreover, $\text{range}(T)$ is a subspace of R^m, $\ker(T)$ is a subspace of R^n, and

$$\text{range}(T) = \mathcal{CS}(A), \quad \text{and} \quad \ker(T) = \mathcal{N}(A).$$

Onto & One-to-One

Theorem

Let $T : R^n \to R^m$ be a linear transformation. Then

1. T is onto if and only if $\dim(\text{range}(T)) = m$, and
2. T is one-to-one if and only if $\dim(\ker(T)) = 0$.

Since $\text{range}(T) = \mathcal{CS}(A)$, T is onto if and only if A has a pivot in every row.

Since $\ker(T) = \mathcal{N}(A)$, T is one-to-one if and only if all columns of A are pivot columns.

Invertible Linear Transformations

Inverse of a Linear Transformation

Let $T : R^n \to R^n$ be a linear transformation with standard matrix A. Then T is invertible if and only if A is an invertible matrix. In this case,

$$T^{-1}(\vec{x}) = A^{-1}\vec{x}$$

for each $\vec{x}$ in R^n.

The standard matrix for T^{-1} is the inverse of the standard matrix for T.

Example

Let $T : R^2 \to R^2$ be the linear transformation given by $T(\langle x_1, x_2\rangle) = \langle x_1 + x_2, x_1 - x_2\rangle$. Show that T is invertible, and find $T^{-1}(\langle x_1, x_2\rangle)$.

$T^{-1}(\vec{x}) = A^{-1}\vec{x}$ if $T(\vec{x}) = A\vec{x}$

Find A: $T(\vec{e}_1) = T(\langle 1,0\rangle) = \langle 1,1\rangle$

$T(\vec{e}_2) = T(\langle 0,1\rangle) = \langle 1,-1\rangle$

$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Find A^{-1}

$[A \mid I_2] = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} & \frac{-1}{2} \end{array}\right]$

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$T^{-1}(\vec{x}) = A^{-1}\vec{x}$$

$$T^{-1}(\langle x_1, x_2\rangle) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \langle x_1, x_2\rangle$$

$$= \left\langle \tfrac{1}{2}x_1 + \tfrac{1}{2}x_2, \tfrac{1}{2}x_1 - \tfrac{1}{2}x_2 \right\rangle$$

$$\boxed{T^{-1}(\langle x_1, x_2\rangle) = \left\langle \tfrac{1}{2}x_1 + \tfrac{1}{2}x_2, \tfrac{1}{2}x_1 - \tfrac{1}{2}x_2 \right\rangle}$$

5.3 Visualizing Linear Transformations

We want to consider certain linear mappings from R^2 to R^2 that correspond to geometric transformations.

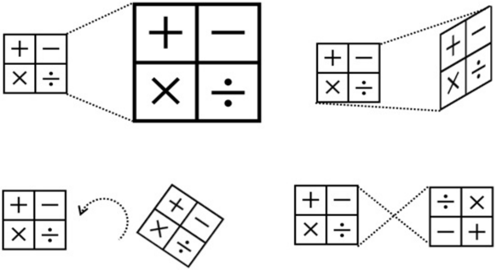


Figure: Scaling, shearing, rotations, reflections

Scaling Transformation

Let $r > 0$ and define $T : R^2 \longrightarrow R^2$ by $T(\vec{x}) = r\vec{x}$.
T is a **dilation** if $r > 1$ and a **contraction** if $0 < r < 1$.

Find the standard matrix of T.

$T(\vec{e}_1) = r\vec{e}_1 = r\langle 1,0\rangle = \langle r,0\rangle$

$T(\vec{e}_2) = r\vec{e}_2 = r\langle 0,1\rangle = \langle 0,r\rangle$

$$A = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} = rI_2$$

The Geometry of Dilation/Contraction

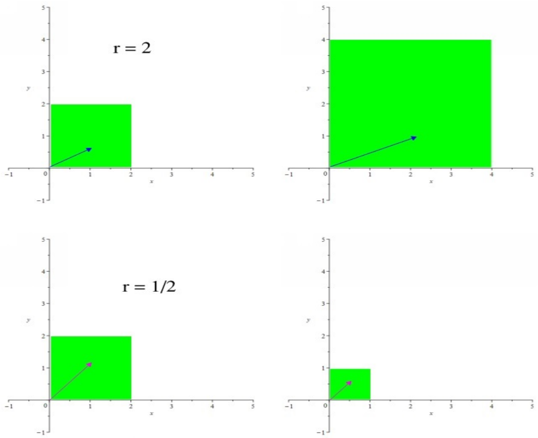


Figure: The 2×2 square in the plane under the dilation $\vec{x} \mapsto 2\vec{x}$ (top) and the contraction $\vec{x} \mapsto \frac{1}{2}\vec{x}$ (bottom). Each includes an example of a single vector and its image.

A Shear Transformation on R^2

Find the standard matrix for the linear transformation from $R^2 \to R^2$ that maps $\vec{e}_2$ to $\vec{e}_1 + \vec{e}_2$ and leaves $\vec{e}_1$ unchanged.

Calling the shear S,

$$S(\vec{e}_1) = S(\vec{e}_1) = \vec{e}_1 = \langle 1,0 \rangle$$

$$S(\vec{e}_2) = \vec{e}_1 + \vec{e}_2 = \langle 1,0 \rangle + \langle 0,1 \rangle = \langle 1,1 \rangle$$

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

A Shear Transformation on R^2

For the shear transformation with standard matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ find the images of the vectors

$$\vec{x}_1 = \langle 0, -1 \rangle, \quad \vec{x}_2 = \langle 1, -1 \rangle, \quad \text{and} \quad \vec{x}_3 = \langle 1, 1 \rangle$$

$$S(\vec{x}_1) = S(\langle 0, -1 \rangle) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \langle 0, -1 \rangle = \langle -1, -1 \rangle$$

$$S(\vec{x}_2) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \langle 1, -1 \rangle = \langle 0, -1 \rangle$$

$$S(\vec{x}_3) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \langle 1, 1 \rangle = \langle 2, 1 \rangle$$

A Shear Transformation on R^2

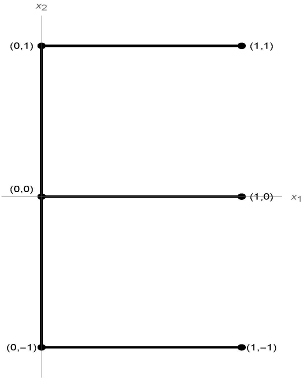
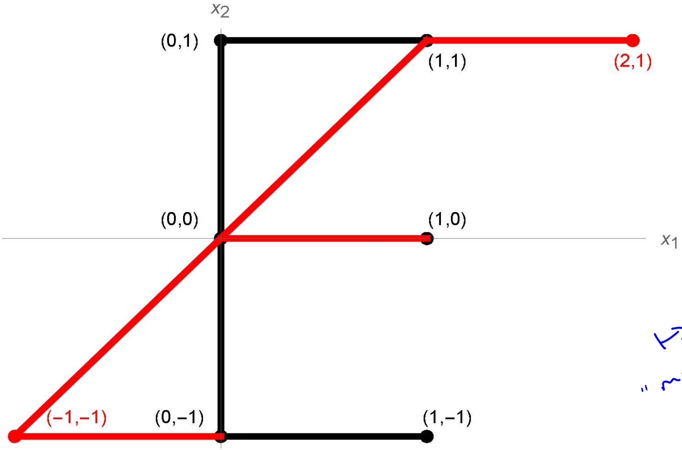


Figure: The letter "E" from line segments connecting select points from $\{(0,-1),(0,0),(0,1),(1,-1),(1,0),(1,1)\}$.

A Shear Transformation on R^2



$\mapsto$ " maps to "

Figure: Letter "E" mapped under the shear transformation $\vec{x} \mapsto \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \vec{x}$.

Shearing Transformations

The matrix for a shearing transformation looks like one of

$$\underbrace{\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}}_{\text{horizontal shear}} \quad \text{or} \quad \underbrace{\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}}_{\text{vertical shear}}$$

(A shear matrix is what you get from I_2 by doing one row replacement: $kR_i + R_j \to R_j$.)

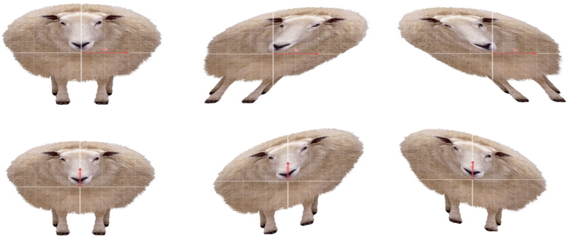


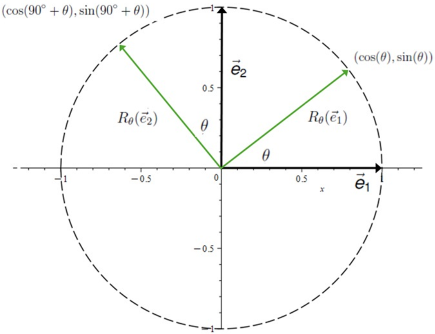
Figure: **Sheared Sheep** (Top) Horizontal Shear (right $k > 0$ and left $k < 0$), (Bottom) Vertical Shear (up $k > 0$ and down $k < 0$).

A Rotation on R^2

Let $R_\theta : R^2 \longrightarrow R^2$ be the rotation transformation that rotates each point in R^2 counter clockwise about the origin through an angle θ. Find the standard matrix for R_θ.

$R_\theta(\vec{e}_1) = \langle \cos\theta, \sin\theta \rangle$

$R_\theta(\vec{e}_2) = \langle \cos(90^\circ+\theta), \sin(90^\circ+\theta) \rangle$



$\cos(90^\circ+\theta) = \cos 90^\circ \cos\theta - \sin 90^\circ \sin\theta$
$= -\sin\theta$

$\sin(90^\circ+\theta) = \sin 90^\circ \cos\theta + \cos 90^\circ \sin\theta$
$= \cos\theta$

$R_\theta(\vec{e}_2) = \langle -\sin\theta, \cos\theta \rangle$

$$A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

A Rotation in R^2

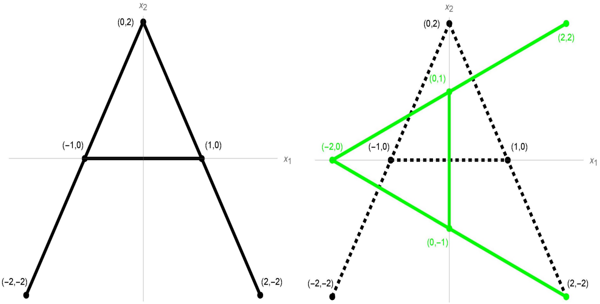
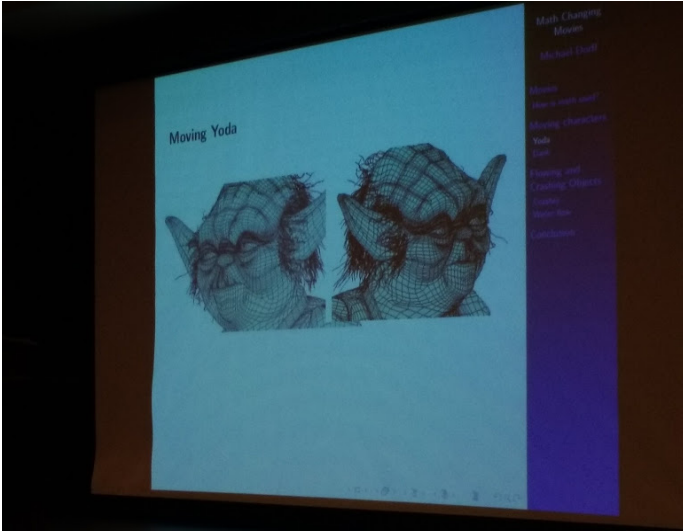


Figure: The letter “A” under a rotation transformation R_{90°.

The standard matrix

$$A_{90^\circ} = \begin{bmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Rotation in Animation



Rotation in Animation