

Section 17: Fourier Series: Trigonometric Series

Suppose the function f is piecewise continuous on the interval $(-p, p)$ for some $p > 0$. We wish to express f as a **Fourier Series**

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \left(\frac{n\pi x}{p} \right) + b_n \sin \left(\frac{n\pi x}{p} \right) \right)$$

The Fourier Series of $f(x)$ on (p, p)

For f defined on $(-p, p)$,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \left(\frac{n\pi x}{p} \right) + b_n \sin \left(\frac{n\pi x}{p} \right) \right)$$

where

$$a_0 = \frac{1}{p} \int_{-p}^p f(x) dx,$$

$$a_n = \frac{1}{p} \int_{-p}^p f(x) \cos \left(\frac{n\pi x}{p} \right) dx, \quad \text{and}$$

$$b_n = \frac{1}{p} \int_{-p}^p f(x) \sin \left(\frac{n\pi x}{p} \right) dx$$

Note: This the most general set of formulas. The case $p = \pi$ is just an example.

The Fourier Series of $f(x)$ on $(-\pi, \pi)$

The **Fourier series** of the function f defined on $(-\pi, \pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

Where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx,$$

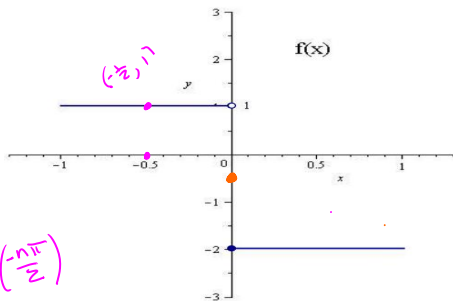
$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx, \quad \text{and}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$$

Example

$$f(x) = \begin{cases} 1, & -1 < x < 0 \\ -2, & 0 \leq x < 1 \end{cases}$$

$$p = 1 \quad \frac{n\pi x}{p} = n\pi x$$



$$\frac{1 + (-2)}{2} = -\frac{1}{2}$$

$$-\frac{1}{2} + \sum_{n=1}^{\infty} \frac{3((-1)^n - 1)}{n\pi} \sin\left(\frac{n\pi x}{2}\right)$$

We found the series
$$f(x) = -\frac{1}{2} + \sum_{n=1}^{\infty} \frac{3((-1)^n - 1)}{n\pi} \sin(n\pi x).$$

Convergence?

The last example gave the series

$$f(x) = -\frac{1}{2} + \sum_{n=1}^{\infty} \frac{3((-1)^n - 1)}{n\pi} \sin(n\pi x).$$

This example raises an interesting question: The function f is not continuous on the interval $(-1, 1)$. However, each term in the Fourier series, and any partial sum thereof, is obviously continuous. This raises questions about properties (e.g. continuity) of the series. More to the point, we may ask: *what is the connection between f and its Fourier series at the point of discontinuity?*

This is the convergence issue mentioned earlier.

Convergence of the Series

Theorem: If f is continuous at x_0 in $(-p, p)$, then the series converges to $f(x_0)$ at that point. If f has a jump discontinuity at the point x_0 in $(-p, p)$, then the series **converges in the mean** to the average value

$$\frac{f(x_0-) + f(x_0+)}{2} \stackrel{\text{def}}{=} \frac{1}{2} \left(\lim_{x \rightarrow x_0^-} f(x) + \lim_{x \rightarrow x_0^+} f(x) \right)$$

at that point.

Periodic Extension:

The series is also defined for x outside of the original domain $(-p, p)$. The extension to all real numbers is $2p$ -periodic.

Convergence of the Series

$$f(x) = \begin{cases} 1, & -1 < x < 0 \\ -2, & 0 \leq x < 1 \end{cases}, \quad f(x) = -\frac{1}{2} + \sum_{n=1}^{\infty} \frac{3((-1)^n - 1)}{n\pi} \sin(n\pi x).$$

-f

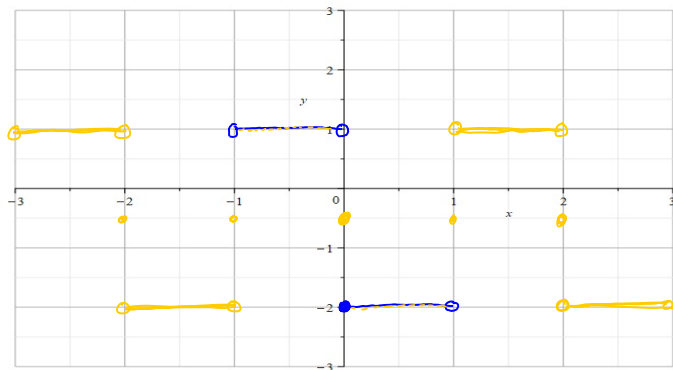


Figure: Plot of the infinite sum, the limit for the Fourier series of f .

Convergence of the Series

$$f(x) = \begin{cases} 1, & -1 < x < 0 \\ -2, & 0 \leq x < 1 \end{cases}, \quad f(x) = -\frac{1}{2} + \sum_{n=1}^{\infty} \frac{3((-1)^n - 1)}{n\pi} \sin(n\pi x).$$

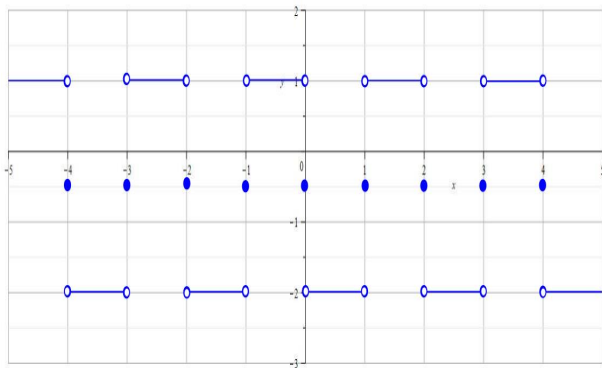


Figure: Plot of the infinite sum, the limit for the Fourier series of f .

The p -series $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Let $f(x) = x^2$ for $-\pi < x < \pi$. With a little effort, it can be shown that f has the Fourier series

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos(nx).$$

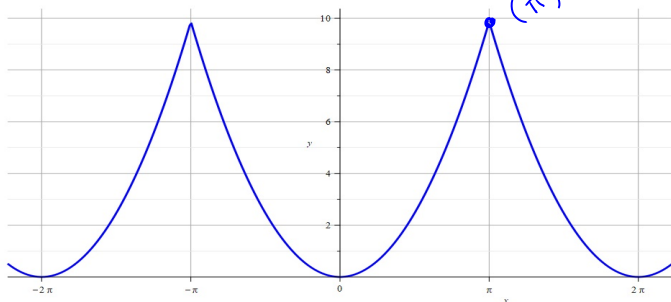


Figure: Series for $f(x) = x^2$, $-\pi < x < \pi$ showing periodic extension.

The p -series $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Let's label this series $S(x)$. $S(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos(nx)$.

Let's evaluate $S(\pi)$

$$S(\pi) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos(n\pi)$$

$$\pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos(n\pi)$$

$$\cos(n\pi) = (-1)^n, \quad (-1)^n (-1)^n = (-1)^{2n} = ((-1)^2)^n = 1^n$$

$$\pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}$$

$$\Rightarrow \frac{2\pi^2}{3} = 4 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\Rightarrow \frac{\frac{2\pi^2}{3}}{4} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Find the Fourier Series for $f(x) = x$, $-1 < x < 1$

$$P = 1 \quad \frac{n\pi x}{P} = \frac{n\pi x}{1} = n\pi x$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) + b_n \sin(n\pi x)$$

Find a 's + b 's.

$$a_0 = \frac{1}{1} \int_{-1}^1 f(x) dx = \int_{-1}^1 x dx = \left. \frac{x^2}{2} \right|_{-1}^1 = \frac{1^2}{2} - \frac{(-1)^2}{2} = 0$$

$$a_0 = 0$$

$$a_n = \frac{1}{1} \int_{-1}^1 f(x) \cos(n\pi x) dx = \int_{-1}^1 x \cos(n\pi x) dx$$

Int by parts $u=x, dv=dx$

$$dv = \cos(n\pi x) dx, \quad v = \frac{1}{n\pi} \sin(n\pi x)$$

$$a_n = \frac{x}{n\pi} \sin(n\pi x) \Big|_{-1}^1 - \int_{-1}^1 \frac{1}{n\pi} \sin(n\pi x) dx$$

$$= \frac{x}{n\pi} \sin(n\pi x) \Big|_{-1}^1 + \frac{1}{n^2\pi^2} \cos(n\pi x) \Big|_{-1}^1$$

0

$$= \frac{1}{n^2\pi^2} \cos(n\pi) - \frac{1}{n^2\pi^2} \cos(-n\pi) = 0$$

recall $\cos(-n\pi) = \cos(n\pi)$

$$a_n = 0 \text{ for all } n=1, 2, 3, \dots$$

$$b_n = \frac{1}{1} \int_{-1}^1 f(x) \sin(n\pi x) dx$$

$$= \int_{-1}^1 x \sin(n\pi x) dx$$

$$= \frac{-x}{n\pi} \cos(n\pi x) \Big|_{-1}^1 + \int_{-1}^1 \frac{1}{n\pi} \cos(n\pi x) dx$$

$$= \frac{-1}{n\pi} \cos(n\pi) - \frac{+(-1)}{n\pi} \cos(-n\pi)$$

$$= -\frac{2}{n\pi} (-1)^n$$

$$b_n = \frac{2}{n\pi} (-1)^{n+1}$$

Int by parts

$$u = x, \quad du = dx$$

$$dv = \sin(n\pi x) dx$$

$$v = \frac{-1}{n\pi} \cos(n\pi x)$$

$$\cos(n\pi) = (-1)^n$$

$$-2(-1)^n = 2(-1)(-1)^n = 2(-1)^{n+1}$$

Using $a_n = 0$ for all $n \geq 0$,

$$f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi x)$$

for $f(x) = x$ on $-1 < x < 1$

Symmetry

For $f(x) = x$, $-1 < x < 1$

$$f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi x)$$

Observation: f is an odd function. It is not surprising then that there are no nonzero constant or cosine terms (which have even symmetry) in the Fourier series for f .

The following plots show f , f plotted along with some partial sums of the series, and f along with a partial sum of its series extended outside of the original domain $(-1, 1)$.

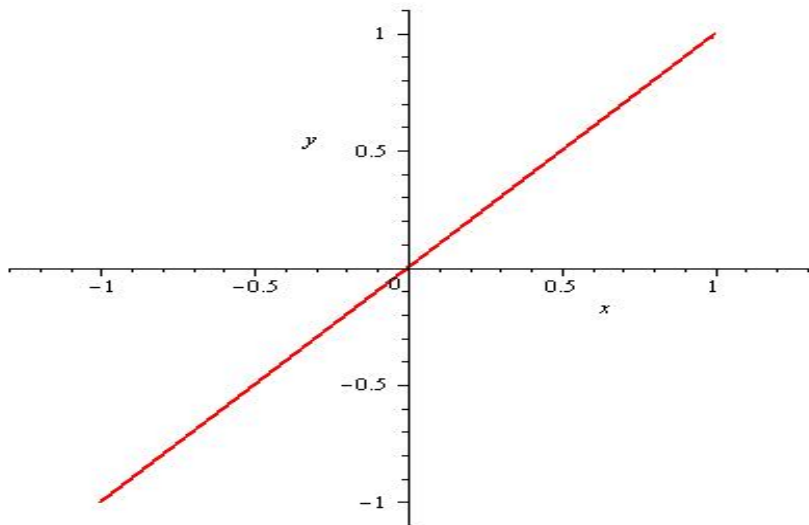


Figure: Plot of $f(x) = x$ for $-1 < x < 1$

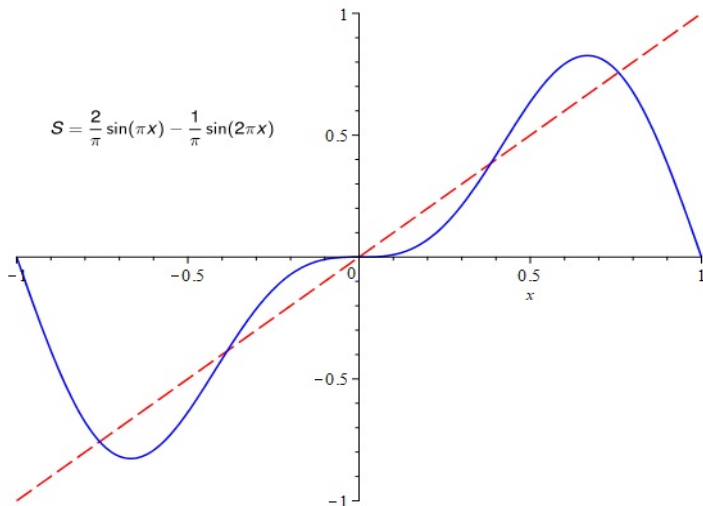


Figure: Plot of $f(x) = x$ for $-1 < x < 1$ with two terms of the Fourier series.

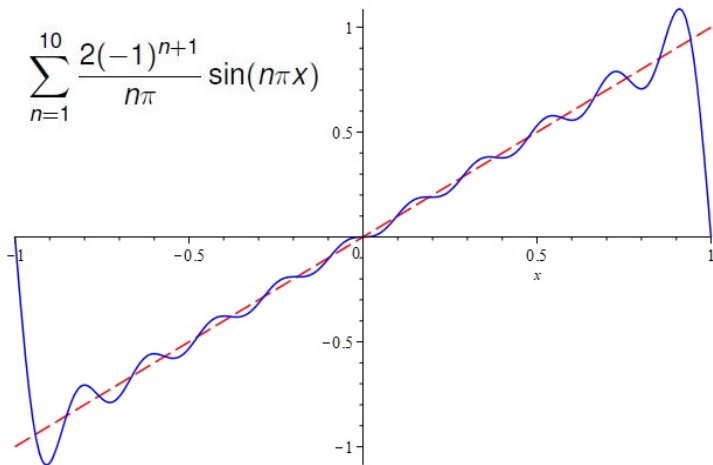


Figure: Plot of $f(x) = x$ for $-1 < x < 1$ with 10 terms of the Fourier series

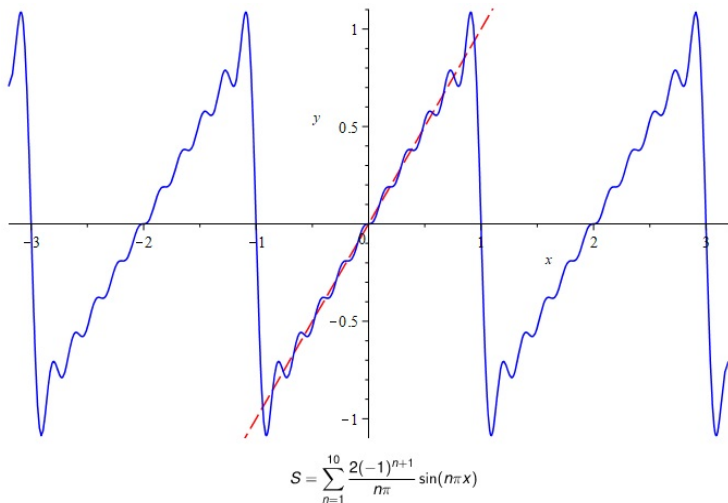


Figure: Plot of $f(x) = x$ for $-1 < x < 1$ with the Fourier series plotted on $(-3, 3)$. Note that the series repeats the profile every 2 units. At the jumps, the series converges to $(-1 + 1)/2 = 0$.

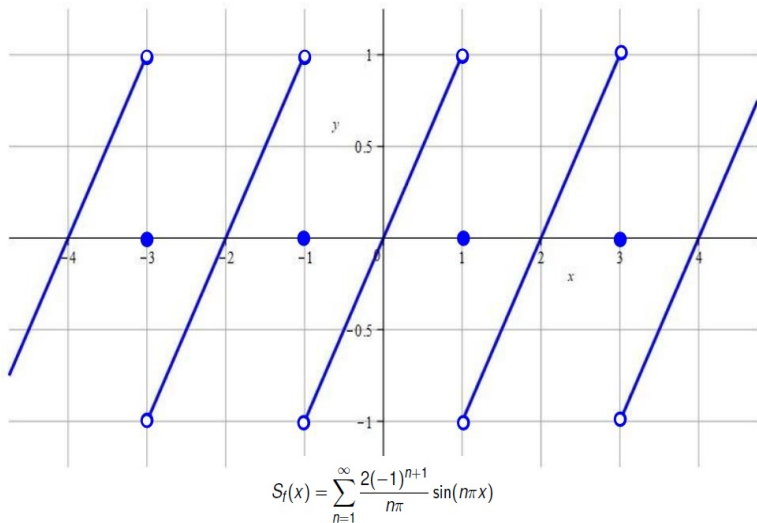


Figure: Plot of $f(x) = x$ for $-1 < x < 1$ with the Fourier series plotted on $(-3, 3)$. Note that the series repeats the profile every 2 units. At the jumps, the series converges to $(-1 + 1)/2 = 0$.

Section 18: Sine and Cosine Series

Functions with Symmetry

Recall some definitions:

Suppose f is defined on an interval containing x and $-x$.

If $f(-x) = f(x)$ for all x , then f is said to be **even**.

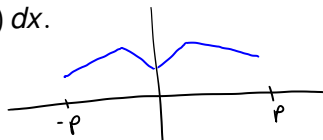
If $f(-x) = -f(x)$ for all x , then f is said to be **odd**.

For example, $f(x) = x^n$ is even if n is even and is odd if n is odd. The trigonometric function $g(x) = \cos x$ is even, and $h(x) = \sin x$ is odd.

Integrals on symmetric intervals

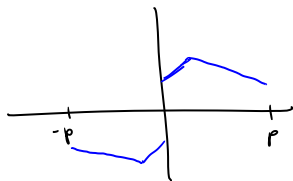
If f is an even function on $(-p, p)$, then

$$\int_{-p}^p f(x) dx = 2 \int_0^p f(x) dx.$$



If f is an odd function on $(-p, p)$, then

$$\int_{-p}^p f(x) dx = 0.$$



Products of Even and Odd functions

$$\text{Even} \times \text{Even} = \text{Even},$$

and

$$\text{Odd} \times \text{Odd} = \text{Even}.$$

While

$$\text{Even} \times \text{Odd} = \text{Odd}.$$

So, suppose f **is even** on $(-p, p)$. This tells us that $f(x) \cos(nx)$ is **even** for all n and $f(x) \sin(nx)$ is **odd** for all n .

And, if f **is odd** on $(-p, p)$. This tells us that $f(x) \sin(nx)$ is **even** for all n and $f(x) \cos(nx)$ is **odd** for all n .

Fourier Series of an Even Function

If f is even on $(-p, p)$, then the Fourier series of f has only constant and cosine terms. Moreover

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{p}\right)$$

where

$$a_0 = \frac{2}{p} \int_0^p f(x) dx$$

and

$$a_n = \frac{2}{p} \int_0^p f(x) \cos\left(\frac{n\pi x}{p}\right) dx.$$

Fourier Series of an Odd Function

If f is odd on $(-p, p)$, then the Fourier series of f has only sine terms. Moreover

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{p}\right)$$

where

$$b_n = \frac{2}{p} \int_0^p f(x) \sin\left(\frac{n\pi x}{p}\right) dx.$$