November 15 Math 2306 sec. 52 Spring 2023

Section 15: Shift Theorems

We defined the unit step function and how it can be used to obtain a translated function while keeping it defined on $[0, \infty)$.

Definition: Unit Step Function

Let $a > 0$. The unit step function *centered at a* is denoted $\mathscr{U}(t-a)$. It is defined by

$$\mathscr{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a \end{cases}$$

Given a function $f(t)$ for $t \ge 0$, and a number $a > 0$

$$f(t-a)\mathscr{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ f(t-a), & t \ge a \end{cases}.$$

The Unit Step Function

There are various notations used for the unit step function. Some of the more common include

$$\mathscr{U}(t-a), \quad \mathbf{u}(t-a), \quad \mathbf{u}_a(t), \quad \theta(t-a), \quad \text{and} \quad H(t-a).$$

There are also variations[1] in how it's defined at the point of discontinuity. In the definition given here, we are taking $\mathscr{U}(0) = 1$, which results in the function being continuous from the right but not continuous from the left at $t = a$. Since we're interested in Laplace transforms, and changing an integrand at a single point won't affect the integral, these discrepancies don't really cause us any trouble. We'll take $\mathscr{U}(t)$ (i.e., the case when $a = 0$) to be

$$\mathscr{U}(t) = 1, \quad \text{for all} \quad t \geq 0.$$

[1] The value $\mathscr{U}(0)$ is typically taken to be one of 1, 0, or $\frac{1}{2}$.

Theorem (translation in t)

If $F(s) = \mathscr{L}\{f(t)\}$ and $a > 0$, then

$$\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} = e^{-as}F(s).$$

Equivalently $\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathscr{U}(t-a).$

e.g. $\mathscr{L}\{t^4\} = \frac{4!}{s^5} \implies \mathscr{L}\{(t-3)^4\mathscr{U}(t-3)\} = \frac{4!e^{-3s}}{s^5}.$

$\mathscr{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\} = \sin(2t)$

and $\mathscr{L}^{-1}\left\{\frac{2e^{-\frac{1}{2}s}}{s^2+4}\right\} = \sin\left(2\left(t-\frac{1}{2}\right)\right)\mathscr{U}\left(t-\frac{1}{2}\right).$

Example

Find the Laplace transform $\mathscr{L}\{f(t)\}$ where

$$f(t) = \begin{cases} 1, & 0 \le t < 1 \\ t, & t \ge 1 \end{cases}$$

(a) First write f in terms of unit step functions.

$$f(t) = 1 - 1\,u(t-1) + t\,u(t-1)$$

$$f(t) = 1 + (-1 + t)\,u(t-1)$$

$$f(t) = 1 + (t-1)\,u(t-1)$$

Example Continued...

(b) Now use the fact that $f(t) = 1 + (t-1)\mathscr{U}(t-1)$ to find $\mathscr{L}\{f\}$.

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{1 + (t-1)\mathcal{U}(t-1)\}$$

$$= \mathcal{L}\{1\} + \mathcal{L}\{(t-1)\mathcal{U}(t-1)\}$$

$$= \frac{1}{s} + e^{-1s}\left(\frac{1!}{s^2}\right) = \frac{1}{s} + \frac{e^{-s}}{s^2}$$

If $\hat{f}(t-1) = t-1$ what is $\hat{f}(t)$? it's $\hat{f}(t) = t$

$$\mathcal{L}\{f(t-1)\mathcal{U}(t-1)\} = e^{-1s}\mathcal{L}\{f(t)\}$$

Alternative Form for Translation in t

It is often the case that we wish to take the transform of a product of the form

$$g(t)\mathscr{U}(t-a)$$

in which the function g is not translated.

The main theorem statement

$$\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} = e^{-as}F(s).$$

can be restated as

$$\boxed{\mathscr{L}\{g(t)\mathscr{U}(t-a)\} = e^{-as}\mathscr{L}\{g(t+a)\}.}$$

This is based on the observation that

$$g(t) = g((t+a)-a).$$

Example

$$\mathscr{L}\{g(t)\mathscr{U}(t-a)\} = e^{-as}\mathscr{L}\{g(t+a)\}$$

Example: Find $\mathscr{L}\left\{\cos t\,\mathscr{U}\left(t-\frac{\pi}{2}\right)\right\} = e^{-\frac{\pi}{2}s}\mathscr{L}\left\{\cos\left(t+\frac{\pi}{2}\right)\right\}$

$= e^{-\frac{\pi}{2}s}\mathscr{L}\{-\sin t\}$

$= -e^{-\frac{\pi}{2}s}\mathscr{L}\{\sin t\} = -e^{-\frac{\pi}{2}s}\left(\frac{1}{s^2+1^2}\right)$

$= \dfrac{-e^{-\frac{\pi}{2}s}}{s^2+1}$

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\cos\left(t+\frac{\pi}{2}\right) = \cos t\underbrace{\cos\frac{\pi}{2}}_{0} - \sin t\underbrace{\sin\frac{\pi}{2}}_{1} = -\sin t$

Example

$$\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathscr{U}(t-a)$$

Example: Find $\mathscr{L}^{-1}\left\{\dfrac{e^{-2s}}{s(s+1)}\right\}$

We need $\mathscr{L}^{-1}\left\{\dfrac{1}{s(s+1)}\right\} = f(t)$

we can use the result

$\mathscr{L}^{-1}\{G(s)H(s)\} = (g*h)(t)$

$= \int_0^t g(\tau)h(t-\tau)\,d\tau$

$\dfrac{1}{s(s+1)} = \dfrac{1}{s+1} \cdot \dfrac{1}{s}$

Let $G(s) = \frac{1}{s+1}$, $g(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}$

$H(s) = \frac{1}{s}$, $h(t) = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$

$g(\tau) = e^{-\tau}$, $h(t-\tau) = 1$

$$\int_0^t g(\tau)h(t-\tau)\,d\tau = \int_0^t e^{-\tau}\,d\tau$$

$$= -e^{-\tau}\Big|_0^t = -e^{-t} - (-e^0) = 1 - e^{-t}$$

$$f(t) = 1 - e^{-t}$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s(s+1)}\right\} = f(t-2)\mathcal{U}(t-2)$$

$$= \left(1 - e^{-(t-2)}\right)\mathcal{U}(t-2)$$

Section 16: Laplace Transforms of Derivatives and IVPs

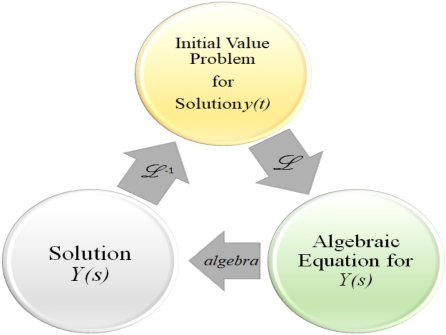

Figure: We'll use the Laplace transform as a tool for solving certain IVPs and systems of IVPs. Our use will be restricted to IVPs with **constant coefficients** and initial conditions given at $t = 0$.

First: Let's look at differentiation.

Transforms of Derivatives

We saw[2] how the following is obtained from the definition of the Laplace transform and a bit of integration by parts.

The Laplace Transform of a Derivative

Suppose f is differentiable on $[0, \infty)$ and $F(s) = \mathscr{L}\{f(t)\}$, then

$$\mathscr{L}\{f'(t)\} = sF(s) - f(0).$$

We can use this result recursively to get transforms for higher order derivatives.

[2] See Worksheet 14 for details.

Transforms of Derivatives

Suppose $F(s) = \mathscr{L}\{f(t)\}$ so that $\mathscr{L}\{f'(t)\} = sF(s) - f(0)$. Express $\mathscr{L}\{f''(t)\}$ in terms of F.

$$\mathscr{L}\{f''(t)\} = s\mathscr{L}\{f'(t)\} - f'(0)$$

$$= s\left(sF(s) - f(0)\right) - f'(0)$$

$$= s^2F(s) - sf(0) - f'(0)$$

Remark

Note that the operation of differentiation where the variable t lives corresponds to an algebraic operation, *multiply by some power of s and add a polynomial*, where s lives.

The Laplace Transform of Derivatives

For $y = y(t)$ defined on $[0, \infty)$ having derivatives y', y'' and so forth, if $\mathscr{L}\{y(t)\} = Y(s)$, then

$$\begin{aligned}
\mathscr{L}\left\{\frac{dy}{dt}\right\} &= sY(s) - y(0)\\
\mathscr{L}\left\{\frac{d^2y}{dt^2}\right\} &= s^2Y(s) - sy(0) - y'(0)\\
\mathscr{L}\left\{\frac{d^3y}{dt^3}\right\} &= s^3Y(s) - s^2y(0) - sy'(0) - y''(0)\\
&\vdots\\
\mathscr{L}\left\{\frac{d^ny}{dt^n}\right\} &= s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - y^{(n-1)}(0).
\end{aligned}$$

Warning about Notation

- The characters Y and y DO NOT represent the same thing. They CANNOT be used interchangeably.

- An expression such as $y'(0)$ means the value of the function $y'(t)$ when the input $t = 0$.

- The function $\mathscr{L}\{y(t)\}$ depends on s NOT on t.

- And, the function $\mathscr{L}^{-1}\{Y(s)\}$ depends on t NOT on s.

Solving and IVP

Use the Laplace transform to solve the initial value problem.

$$y'' - 6y' + 8y = t, \quad y(0) = 1, \quad y'(0) = 2$$

Take $\mathcal{L}$ of both sides of the ODE.

Let $Y(s) = \mathcal{L}\{y(t)\}$.

$$\mathcal{L}\{y'' - 6y' + 8y\} = \mathcal{L}\{t\}$$

$$\mathcal{L}\{y''\} - 6\mathcal{L}\{y'\} + 8\mathcal{L}\{y\} = \frac{1!}{s^2}$$

$$s^2Y(s) - sy(0) - y'(0) - 6\left(sY(s) - y(0)\right) + 8Y(s) = \frac{1}{s^2}$$

sub in $y(0)=1$ and $y'(0)=2$ and isolate $Y(s)$.

$$s^2Y(s) - s(1) - 2 - 6\left(sY(s) - 1\right) + 8Y(s) = \frac{1}{s^2}$$

$$(s^2 - 6s + 8)Y(s) - s - 2 + 6 = \frac{1}{s^2}$$

$$\underbrace{(s^2 - 6s + 8)}_{\text{characteristic polynomial for the ODE}}Y(s) = \frac{1}{s^2} + s - 4$$

$$\boxed{Y(s) = \frac{\frac{1}{s^2}}{s^2 - 6s + 8} + \frac{s-4}{s^2-6s+8}}$$

The solution to the IVP is

$$\mathscr{L}^{-1}\{Y(s)\} = y(t)$$

We'll finish this next time.