November 15 Math 2306 sec. 53 Fall 2024

Section 16: Laplace Transforms of Derivatives and IVPs

Use Laplace Transforms to Solve and IVP

- Start with constant coefficient IVP with IC at $t = 0$. For example,

$$ay'' + by' + cy = g(t), \quad y(0) = y_0, \quad y'(0) = y_1.$$

- Let $Y(s) = \mathscr{L}\{y(t)\}$ and take the transform of both sides of the ODE using any necessary results.
- Sub in the initial conditions where they appear in the transformed equation.
- Use basic algebra to isolate the transform $Y(s)$.
- Using whatever algebra or function identities that are needed, take the inverse transform to obtain the solution

$$y(t) = \mathscr{L}^{-1}\{Y(s)\}.$$

The Laplace Transform of Derivatives

For $y = y(t)$ defined on $[0, \infty)$ having derivatives y', y'' and so forth, if $\mathscr{L}\{y(t)\} = Y(s)$, then

$$\begin{aligned}
\mathscr{L}\left\{\frac{dy}{dt}\right\} &= sY(s) - y(0) \\
\mathscr{L}\left\{\frac{d^2y}{dt^2}\right\} &= s^2Y(s) - sy(0) - y'(0) \\
\mathscr{L}\left\{\frac{d^3y}{dt^3}\right\} &= s^3Y(s) - s^2y(0) - sy'(0) - y''(0) \\
&\vdots \\
\mathscr{L}\left\{\frac{d^ny}{dt^n}\right\} &= s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - y^{(n-1)}(0).
\end{aligned}$$

Solve the IVP using the Laplace Transform

$$y'' + 4y' + 4y = te^{-2t} \quad y(0) = 1, \quad y'(0) = 0$$

Let $Y(s) = \mathcal{L}\{y(t)\}$.

$\mathcal{L}\{te^{-2t}\} = F(s+2)$

when

$F(s) = \mathcal{L}\{t\}$

$= \frac{1!}{s^2}$

$$\mathcal{L}\{y'' + 4y' + 4y\} = \mathcal{L}\{te^{-2t}\}$$

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = \frac{1}{(s+2)^2}$$

$$s^2Y(s) - s\underbrace{y(0)}_{=1} - \underbrace{y'(0)}_{=0} + 4\left(sY(s) - \underbrace{y(0)}_{=1}\right) + 4Y(s) = \frac{1}{(s+2)^2}$$

$$(s^2 + 4s + 4)Y(s) - s - 4 = \frac{1}{(s+2)^2}$$

$$(s^2+4s+4)\,Y(s) = \frac{1}{(s+2)^2} + s + 4$$

Charac. poly

$$y'' + 4y' + 4y = te^{-2t}$$

$$Y(s) = \frac{1}{(s+2)^2(s^2+4s+4)} + \frac{s+4}{s^2+4s+4}$$

Note $s^2+4s+4 = (s+2)^2$

$$Y(s) = \frac{1}{(s+2)^4} + \frac{s+4}{(s+2)^2}$$

For $\frac{s+4}{(s+2)^2} = \frac{s+2+2}{(s+2)^2} = \frac{s+2}{(s+2)^2} + \frac{2}{(s+2)^2} = \frac{1}{s+2} + \frac{2}{(s+2)^2}$

$$Y(s) = \frac{1}{(s+2)^4} + \frac{1}{s+2} + \frac{2}{(s+2)^2}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^4}\right\} = e^{-2t}\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\} = \frac{1}{3!}e^{-2t}\mathcal{L}^{-1}\left\{\frac{3!}{s^4}\right\}$$

$$\frac{1}{s^4}$$

$$= \frac{1}{3!}e^{-2t}t^3$$

The solution to the IVP $y(t) = \mathcal{L}^{-1}\{Y(s)\}$.

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^4} + \frac{1}{s+2} + \frac{2}{(s+2)^2}\right\}$$

$$\boxed{y(t) = \frac{1}{3!}e^{-2t}t^3 + e^{-2t} + 2te^{-2t}}$$

An IVP with Piecewise Input

One of the most powerful uses of the Laplace transform is in applications that involve a piecewise defined forcing function. Let's look at an example.

An LR-series circuit has inductance $L = 1$h, resistance $R = 10\Omega$, and implied voltage $E(t)$ whose graph is given below. If the initial current $i(0) = 0$, find the current $i(t)$ in the circuit.

$$E(t) = \begin{cases} 0, & 0 \le t < 1 \\ E_0, & 1 \le t < 3 \\ 0, & t \geq 3 \end{cases}$$

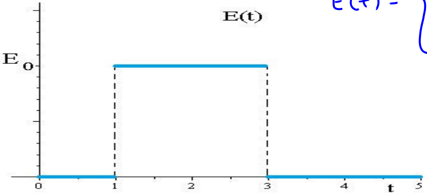

Figure: A switch is closed for two seconds from $t = 1$ until $t = 3$ during which a constant E_0 volts is applied.

$$E(t) = 0 - 0\mathcal{U}(t-1) + E_0\mathcal{U}(t-1) - E_0\mathcal{U}(t-3) + 0\mathcal{U}(t-3)$$

An IVP with Piecewise Input

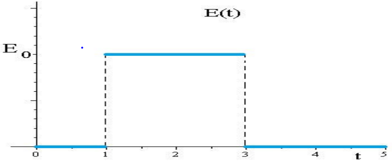

Figure: $L\dfrac{di}{dt} + Ri = E(t)$, $i(0) = 0$ where $L = 1$, $R = 10$ and $E(t)$ is shown.

$$\frac{di}{dt} + 10i = E_0 u(t-1) - E_0 u(t-3)$$

Let $I(s) = \mathcal{L}\{i(t)\}$.

$$\mathcal{L}\{i' + 10i\} = \mathcal{L}\{E_0 u(t-1) - E_0 u(t-3)\}$$

$$\mathcal{L}\{i'\} + 10\mathcal{L}\{i\} = E_0 \mathcal{L}\{u(t-1)\} - E_0 \mathcal{L}\{u(t-3)\}$$

$$sI(s) - \underset{0}{i(0)} + 10I(s) = E_0\frac{e^{-1s}}{s} - E_0\frac{e^{-3s}}{s}$$

$$(s+10)I(s) = \frac{E_0 e^{-s}}{s} - \frac{E_0 e^{-3s}}{s}$$

$$I(s) = \frac{E_0 e^{-s}}{s(s+10)} - \frac{E_0 e^{-3s}}{s(s+10)}$$

PFD $$\frac{E_0}{s(s+10)} = \frac{\frac{E_0}{10}}{s} - \frac{\frac{E_0}{10}}{s+10}$$

$$I(s) = \frac{E_0}{10}\left(\frac{1}{s} - \frac{1}{s+10}\right)e^{-s} - \frac{E_0}{10}\left(\frac{1}{s} - \frac{1}{s+10}\right)e^{-3s}$$

we'll use $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathcal{U}(t-a)$

where $f(t) = \mathcal{L}^{-1}\{F(s)\}$.

Let $f(t) = \mathcal{L}^{-1}\left\{\frac{E_0}{10}\left(\frac{1}{s} - \frac{1}{s+10}\right)\right\} = \frac{E_0}{10}\left(\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+10}\right\}\right)$

$$= \frac{E_0}{10}\left(1 - e^{-10t}\right)$$

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\,u(t-a)$$

$$I(s) = \frac{E_0}{10}\left(\frac{1}{s} - \frac{1}{s+10}\right)e^{-s} - \frac{E_0}{10}\left(\frac{1}{s} - \frac{1}{s+10}\right)e^{-3s}$$

The current $i(t) = \mathcal{L}^{-1}\{I(s)\}$.

$$i(t) = \frac{E_0}{10}\left(1 - e^{-10(t-1)}\right)u(t-1) - \frac{E_0}{10}\left(1 - e^{-10(t-3)}\right)u(t-3)$$

Let's write $i(t)$ in stacked notation.

For $0 \le t < 1$, $u(t-1) = 0$ and $u(t-3) = 0$

$$i(t) = 0$$

For $1 \le t < 3$ $u(t-1) = 1$ and $u(t-3) = 0$

$$i(t) = \frac{E_0}{10}\left(1 - e^{-10(t-1)}\right)$$

For $t \geq 3$, $u(t-1) = 1$ and $u(t-3) = 1$

$$i(t) = \frac{E_0}{10}\left(-e^{-10(t-1)} + e^{-10(t-3)}\right)$$

$$\boxed{i(t) = \begin{cases} 0 & , \quad 0 \le t < 1 \\ \frac{E_0}{10}\left(1 - e^{-10(t-1)}\right) & , \quad 1 \le t < 3 \\ \frac{E_0}{10}\left(e^{-10(t-3)} - e^{-10(t-1)}\right) & , \quad t \geq 3 \end{cases}}$$