November 15 Math 2306 sec. 54 Fall 2021

Section 17: Fourier Series: Trigonometric Series

Suppose the function f is piecewise continuous on the interval $(-p, p)$ for some $p > 0$. We wish to express f as a **Fourier Series**

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{p}\right) + b_n \sin\left(\frac{n\pi x}{p}\right) \right)$$

The Fourier Series of $f(x)$ on (p, p)

For f defined on $(-p, p)$,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{p}\right) + b_n \sin\left(\frac{n\pi x}{p}\right)\right)$$

where

$$\begin{aligned} a_0 &= \frac{1}{p}\int_{-p}^{p} f(x)\,dx, \\ a_n &= \frac{1}{p}\int_{-p}^{p} f(x)\cos\left(\frac{n\pi x}{p}\right)\,dx, \quad \text{and} \\ b_n &= \frac{1}{p}\int_{-p}^{p} f(x)\sin\left(\frac{n\pi x}{p}\right)\,dx \end{aligned}$$

Note: This the most general set of formulas. The case $p = \pi$ is just an example.

The Fourier Series of $f(x)$ on $(-\pi, \pi)$

The **Fourier series** of the function f defined on $(-\pi, \pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \, .$$

Where

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx, \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad \text{and} \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \end{aligned}$$

Example

f is defined on $(-p, p)$

$$f(x) = \begin{cases} 1, & -1 < x < 0 \\ -2, & 0 \le x < 1 \end{cases}$$

$p = 1 \qquad \frac{n\pi x}{p} = n\pi x$

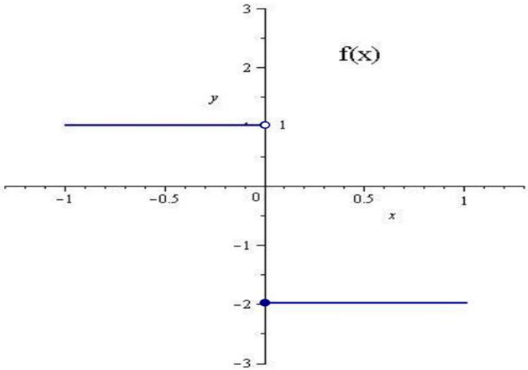


Figure: Let's find the Fourier Series for f.

We need to find the a's & b's.

$$a_0 = \frac{1}{1}\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} 1\,dx + \int_{0}^{1}(-2)\,dx$$

$$= x\Big|_{-1}^{0} + (-2x)\Big|_{0}^{1} = (0-(-1)) + (-2-0) = -1$$

$$\boxed{a_0 = -1}$$

$$a_n = \frac{1}{1}\int_{-1}^{1} f(x)\cos(n\pi x)\,dx$$

$$= \int_{-1}^{0} 1\cos(n\pi x)\,dx + \int_{0}^{1}(-2)\cos(n\pi x)\,dx$$

$$= \frac{1}{n\pi} \sin(n\pi x) \Big|_{-1}^{0} - \frac{2}{n\pi} \sin(n\pi x) \Big|_{0}^{1}$$

$$= \frac{1}{n\pi} \underbrace{\sin(0)}_{=0} - \frac{1}{n\pi} \sin(-n\pi) - \frac{2}{n\pi} \sin(n\pi) + \frac{2}{n\pi} \underbrace{\sin(0)}_{=0}$$

$\sin(n\pi) = 0$ for all integers n.

$$= 0 \qquad \boxed{a_n = 0 \quad \text{for } n = 1, 2, 3, \dots}$$

$$b_n = \frac{1}{1} \int_{-1}^{1} f(x) \sin(n\pi x)\, dx$$

$$= \int_{-1}^{0} 1 \sin(n\pi x)\,dx + \int_{0}^{1} (-2)\sin(n\pi x)\,dx$$

$$= \frac{-1}{n\pi}\cos(n\pi x)\Big|_{-1}^{0} + \frac{2}{n\pi}\cos(n\pi x)\Big|_{0}^{1}$$

$$= \frac{-1}{n\pi}\cos(0) - \frac{-1}{n\pi}\cos(-n\pi) + \frac{2}{n\pi}\cos(n\pi) - \frac{2}{n\pi}\cos(0)$$

$$\cos(-n\pi) = \cos(n\pi) = (-1)^n$$

$$b_n = \frac{-3}{n\pi} + \frac{3}{n\pi}(-1)^n = \frac{3\left((-1)^n - 1\right)}{n\pi}$$

Recall $a_0 = -1$, $a_n = 0$ for $n \geq 1$

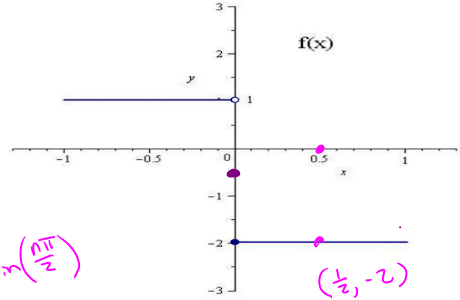
The series has the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) + b_n \sin(n\pi x)$$

$$\boxed{f(x) = \frac{-1}{2} + \sum_{n=1}^{\infty} \frac{3\left((-1)^n - 1\right)}{n\pi} \sin(n\pi x)}$$

Example

$$f(x) = \begin{cases} 1, & -1 < x < 0 \\ -2, & 0 \le x < 1 \end{cases} \qquad p = 1 \quad \frac{n\pi x}{p} = n\pi x$$



$$\frac{1+(-2)}{2} = \frac{-1}{2}$$

$(\frac{1}{2}, -2)$

$$\frac{-1}{2} + \sum_{n=1}^{\infty} \frac{3((-1)^n - 1)}{n\pi} \sin\left(\frac{n\pi}{2}\right)$$

We found the series $\quad f(x) = -\frac{1}{2} + \sum_{n=1}^{\infty} \frac{3((-1)^n - 1)}{n\pi} \sin(n\pi x).$

Convergence?

The last example gave the series

$$f(x) = -\frac{1}{2} + \sum_{n=1}^{\infty} \frac{3((-1)^n - 1)}{n\pi} \sin(n\pi x).$$

This example raises an interesting question: The function f is not continuous on the interval $(-1, 1)$. However, each term in the Fourier series, and any partial sum thereof, is obviously continuous. This raises questions about properties (e.g. continuity) of the series. More to the point, we may ask: *what is the connection between f and its Fourier series at the point of discontinuity?*

This is the convergence issue mentioned earlier.

Convergence of the Series

Theorem: If f is continuous at x_0 in $(-p, p)$, then the series converges to $f(x_0)$ at that point. If f has a jump discontinuity at the point x_0 in $(-p, p)$, then the series **converges in the mean** to the average value

$$\frac{f(x_0-)+f(x_0+)}{2} \stackrel{\text{def}}{=} \frac{1}{2}\left(\lim_{x\to x_0^-} f(x) + \lim_{x\to x_0^+} f(x)\right)$$

at that point.

Periodic Extension:
The series is also defined for x outside of the original domain $(-p, p)$. The extension to all real numbers is $2p$-periodic.

Convergence of the Series

$$f(x) = \begin{cases} 1, & -1 < x < 0 \\ -2, & 0 \le x < 1 \end{cases}, \quad f(x) = -\frac{1}{2} + \sum_{n=1}^{\infty} \frac{3((-1)^n - 1)}{n\pi} \sin(n\pi x).$$

- f(x)

- Series

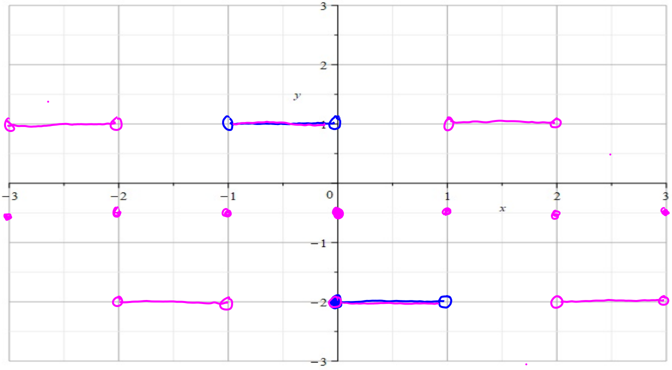


Figure: Plot of the infinite sum, the limit for the Fourier series of f.

Convergence of the Series

$$f(x) = \begin{cases} 1, & -1 < x < 0 \\ -2, & 0 \le x < 1 \end{cases}, \qquad f(x) = -\frac{1}{2} + \sum_{n=1}^{\infty} \frac{3((-1)^n - 1)}{n\pi} \sin(n\pi x).$$

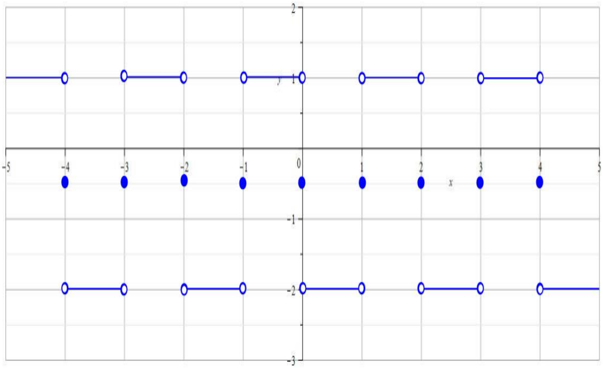


Figure: Plot of the infinite sum, the limit for the Fourier series of f.

Find the Fourier Series for $f(x) = x$, $-1 < x < 1$

$p = 1$, $\quad \frac{n\pi x}{p} = \frac{n\pi x}{1} = n\pi x$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) + b_n \sin(n\pi x)$$

$$a_0 = \frac{1}{1}\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} x\,dx = \frac{x^2}{2}\Big|_{-1}^{1} = \frac{1^2}{2} - \frac{(-1)^2}{2} = 0$$

$$\boxed{a_0 = 0}$$

$$a_n = \frac{1}{1}\int_{-1}^{1} f(x)\cos(n\pi x)\,dx = \int_{-1}^{1} x\cos(n\pi x)\,dx$$

Symmetry

For $f(x) = x, \quad -1 < x < 1$

$$f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi x)$$

Observation: f is an odd function. It is not surprising then that there are no nonzero constant or cosine terms (which have even symmetry) in the Fourier series for f.

The following plots show f, f plotted along with some partial sums of the series, and f along with a partial sum of its series extended outside of the original domain $(-1, 1)$.

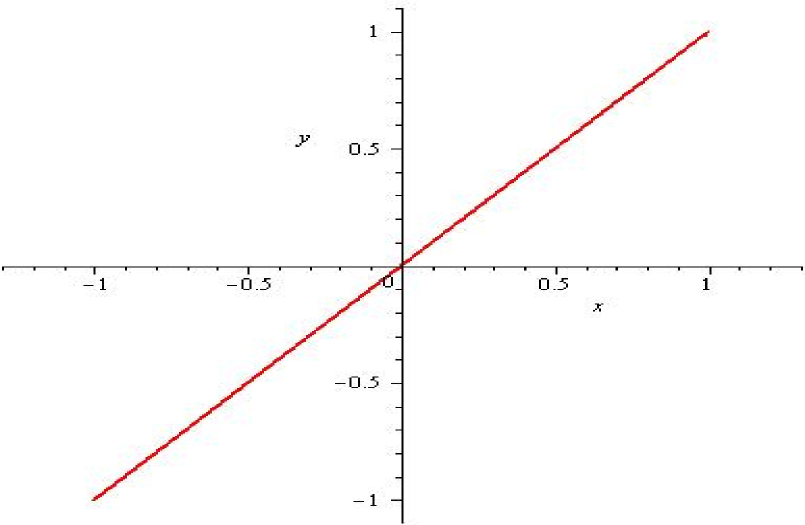


Figure: Plot of $f(x) = x$ for $-1 < x < 1$

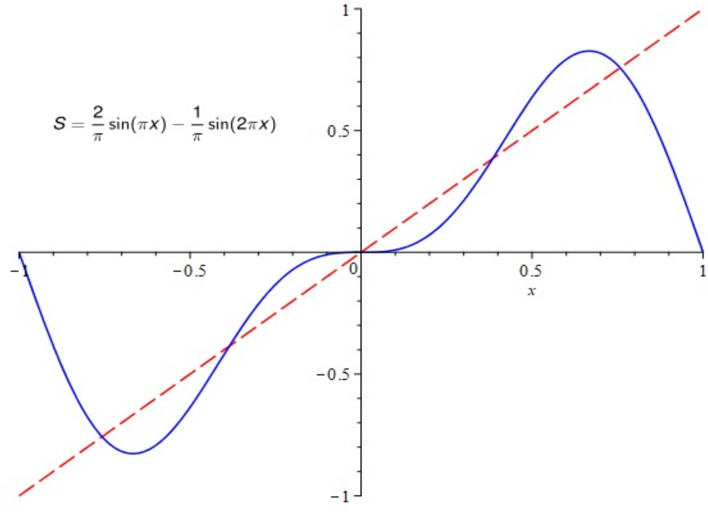


Figure: Plot of $f(x) = x$ for $-1 < x < 1$ with two terms of the Fourier series.

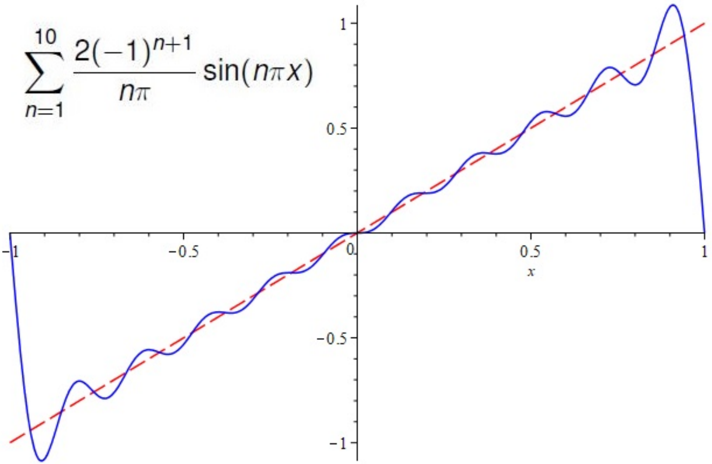


Figure: Plot of $f(x) = x$ for $-1 < x < 1$ with 10 terms of the Fourier series

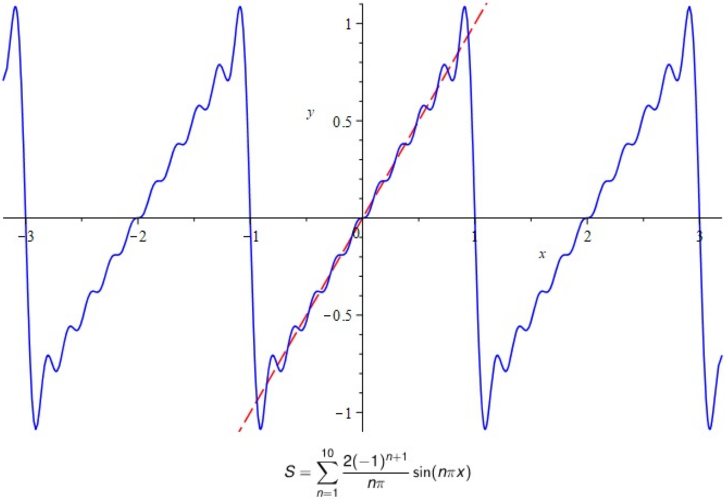


Figure: Plot of $f(x) = x$ for $-1 < x < 1$ with the Fourier series plotted on $(-3, 3)$. Note that the series repeats the profile every 2 units. At the jumps, the series converges to $(-1+1)/2 = 0$.

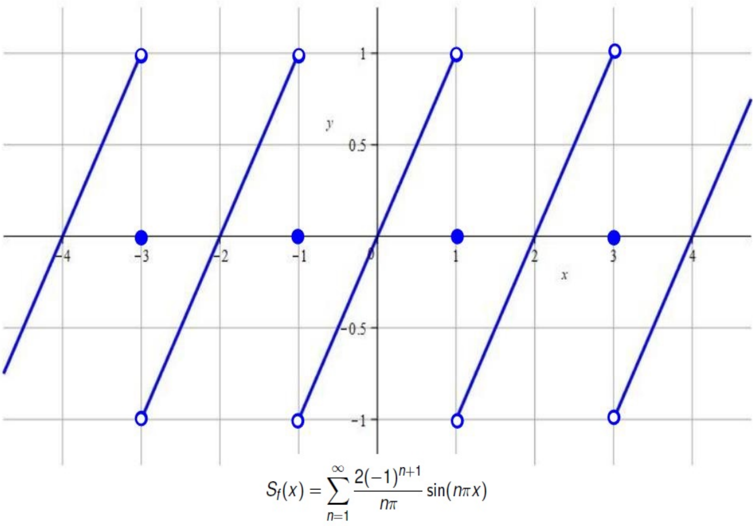


Figure: Plot of $f(x) = x$ for $-1 < x < 1$ with the Fourier series plotted on $(-3, 3)$. Note that the series repeats the profile every 2 units. At the jumps, the series converges to $(-1+1)/2 = 0$.

Functions with Symmetry

Recall some definitions:

Suppose f is defined on an interval containing x and $-x$.

If $f(-x) = f(x)$ for all x, then f is said to be **even.**

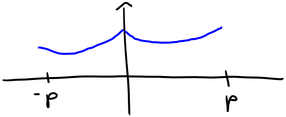
If $f(-x) = -f(x)$ for all x, then f is said to be **odd.**

For example, $f(x) = x^n$ is even if n is even and is odd if n is odd. The trigonometric function $g(x) = \cos x$ is even, and $h(x) = \sin x$ is odd.

Integrals on symmetric intervals

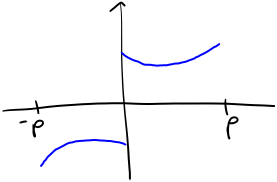
If f is an even function on $(-p, p)$, then

$$\int_{-p}^{p} f(x)\,dx = 2\int_{0}^{p} f(x)\,dx.$$



If f is an odd function on $(-p, p)$, then

$$\int_{-p}^{p} f(x)\,dx = 0.$$



Products of Even and Odd functions

$$\text{Even} \quad \times \quad \text{Even} \quad = \quad \text{Even},$$

and

$$\text{Odd} \quad \times \quad \text{Odd} \quad = \quad \text{Even}.$$

While

$$\text{Even} \quad \times \quad \text{Odd} \quad = \quad \text{Odd}.$$

So, suppose f **is even** on $(-p, p)$. This tells us that $f(x)\cos(nx)$ is even for all n and $f(x)\sin(nx)$ is odd for all n.

And, if f **is odd** on $(-p, p)$. This tells us that $f(x)\sin(nx)$ is even for all n and $f(x)\cos(nx)$ is odd for all n