

## Section 16: Laplace Transforms of Derivatives and IVPs

Let's solve some IVPs using the Laplace transform.

$$y'' + 4y' + 3y = \begin{cases} 0, & 0 \leq t < 6 \\ 6, & 6 \leq t \end{cases} \quad y(0) = -1, \quad y'(0) = -1$$

Rewrite the ODE using the step function.

Rewrite the ODE.

$$y'' + 4y' + 3y = 6\mathcal{U}(t - 6) \quad y(0) = -1, \quad y'(0) = -1$$

Take the transform and apply the I.C.

$$y'' + 4y' + 3y = 6\mathcal{U}(t - 6) \quad y(0) = -1, \quad y'(0) = -1$$

$$s^2 Y(s) - sy(0) - y'(0) + 4(sY(s) - y(0)) + 3Y(s) = \frac{6e^{-6s}}{s} \implies$$

$$s^2 Y(s) + s + 1 + 4sY(s) + 4 + 3Y(s) = \frac{6e^{-6s}}{s}$$

Isolate  $Y(s)$ .

$$s^2 Y(s) + s + 1 + 4sY(s) + 4 + 3Y(s) = \frac{6e^{-6s}}{s}$$

$$Y(s) = \frac{6e^{-6s}}{s(s^2 + 4s + 3)} - \frac{s + 5}{s^2 + 4s + 3} \implies$$

$$Y(s) = \frac{6e^{-6s}}{s(s + 1)(s + 3)} - \frac{s + 5}{(s + 1)(s + 3)}$$

Decompose as needed, and take the Inverse Transform.

$$Y(s) = \frac{6e^{-6s}}{s(s+1)(s+3)} - \frac{s+5}{(s+1)(s+3)}$$

$$Y(s) = e^{-6s} \left( \frac{2}{s} - \frac{3}{s+1} + \frac{1}{s+3} \right) - \frac{2}{s+1} + \frac{1}{s+3}$$

The solution  $y(t) = \mathcal{L}^{-1}\{Y(s)\}$

$$y(t) = \left( 2 - 3e^{-(t-6)} + e^{-3(t-6)} \right) \mathcal{U}(t-6) - 2e^{-t} + e^{-3t}$$

# Solving a System

Consider the system of equations<sup>1</sup>

$$\begin{aligned}\frac{dx}{dt} + y &= 4, & x(0) = 0 \\ -4x + \frac{dy}{dt} + 4y &= 0, & y(0) = 0\end{aligned}$$

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<sup>1</sup>See Worksheet 11 which includes a derivation of this system from a LRC circuit network.

Take the transform and apply the IC.

$$\begin{aligned}\frac{dx}{dt} + y &= 4, & x(0) = 0 \\ -4x + \frac{dy}{dt} + 4y &= 0, & y(0) = 0\end{aligned}$$

$$\begin{aligned}sX(s) + Y(s) &= \frac{4}{s} \\ -4X(s) + (s+4)Y(s) &= 0\end{aligned}$$

Isolate  $X$  and  $Y$ .

$$\begin{bmatrix} s & 1 \\ -4 & s+4 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 4/s \\ 0 \end{bmatrix}$$

$$X(s) = \frac{4(s+4)}{s(s+2)^2}, \quad \text{and} \quad Y(s) = \frac{16}{s(s+2)^2}$$

Decompose and take the inverse transform.

$$X(s) = \frac{4(s+4)}{s(s+2)^2}, \quad \text{and} \quad Y(s) = \frac{16}{s(s+2)^2}$$

$$X(s) = \frac{4}{s} - \frac{4}{s+2} - \frac{4}{(s+2)^2}, \quad \text{and}$$

$$Y(s) = \frac{4}{s} - \frac{4}{s+2} - \frac{8}{(s+2)^2}.$$

$$x(t) = 4 - 4e^{-2t} - 4te^{-2t}, \quad \text{and} \quad y(t) = 4 - 4e^{-2t} - 8te^{-2t}$$