November 16 Math 2306 sec. 52 Fall 2022

Section 16: Laplace Transforms of Derivatives and IVPs

Let's solve some IVPs using the Laplace transform.

$$y'' + 4y' + 3y = \begin{cases} 0, & 0 \le t < 6 \\ 6, & 6 \le t \end{cases} \qquad y(0) = -1, \quad y'(0) = -1$$

Rewrite the ODE using the step function.

Rewrite the ODE.

$$y'' + 4y' + 3y = 6\mathscr{U}(t-6) \quad y(0) = -1, \quad y'(0) = -1$$

Take the transform and apply the I.C.

$$y'' + 4y' + 3y = 6\mathscr{U}(t-6) \quad y(0) = -1, \quad y'(0) = -1$$

$$s^2 Y(s) - sy(0) - y'(0) + 4(sY(s) - y(0)) + 3Y(s) = \frac{6e^{-6s}}{s} \quad \Longrightarrow$$

$$s^2 Y(s) + s + 1 + 4sY(s) + 4 + 3Y(s) = \frac{6e^{-6s}}{s}$$

Isolate $Y(s)$.

$$s^2 Y(s) + s + 1 + 4sY(s) + 4 + 3Y(s) = \frac{6e^{-6s}}{s}$$

$$Y(s) = \frac{6e^{-6s}}{s(s^2+4s+3)} - \frac{s+5}{s^2+4s+3} \quad \Longrightarrow$$

$$Y(s) = \frac{6e^{-6s}}{s(s+1)(s+3)} - \frac{s+5}{(s+1)(s+3)}$$

Decompose as needed, and take the Inverse Transform.

$$Y(s) = \frac{6e^{-6s}}{s(s+1)(s+3)} - \frac{s+5}{(s+1)(s+3)}$$

$$Y(s) = e^{-6s}\left(\frac{2}{s} - \frac{3}{s+1} + \frac{1}{s+3}\right) - \frac{2}{s+1} + \frac{1}{s+3}$$

The solution $y(t) = \mathscr{L}^{-1}\{Y(s)\}$

$$\boxed{y(t) = \left(2 - 3e^{-(t-6)} + e^{-3(t-6)}\right)\mathscr{U}(t-6) - 2e^{-t} + e^{-3t}}$$

Solving a System

Consider the system of equations[1]

$$\begin{array}{rcrcrcl} \dfrac{dx}{dt} & + & y & & & = & 4, \qquad x(0) = 0 \\ -4x & + & \dfrac{dy}{dt} & + & 4y & = & 0, \qquad y(0) = 0 \end{array}$$

[1] See Worksheet 11 which includes a derivation of this system from a LRC circuit network.

Take the transform and apply the IC.

$$\begin{array}{rcrcrcl} \dfrac{dx}{dt} & + & y & & & = & 4, \qquad x(0)=0 \\ -4x & + & \dfrac{dy}{dt} & + & 4y & = & 0, \qquad y(0)=0 \end{array}$$

$$\begin{array}{rcrcl} sX(s) & + & Y(s) & = & \dfrac{4}{s} \\ -4X(s) & + & (s+4)Y(s) & = & 0 \end{array}$$

Isolate X and Y.

$$\begin{bmatrix} s & 1 \\ -4 & s+4 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 4/s \\ 0 \end{bmatrix}$$

$$X(s) = \frac{4(s+4)}{s(s+2)^2}, \quad \text{and} \quad Y(s) = \frac{16}{s(s+2)^2}$$

Decompose and take the inverse transform.

$$X(s) = \frac{4(s+4)}{s(s+2)^2}, \quad \text{and} \quad Y(s) = \frac{16}{s(s+2)^2}$$

$$X(s) = \frac{4}{s} - \frac{4}{s+2} - \frac{4}{(s+2)^2}, \quad \text{and}$$

$$Y(s) = \frac{4}{s} - \frac{4}{s+2} - \frac{8}{(s+2)^2}.$$

$$\boxed{x(t) = 4 - 4e^{-2t} - 4te^{-2t}, \quad \text{and} \quad y(t) = 4 - 4e^{-2t} - 8te^{-2t}}$$