November 17 Math 2306 sec. 51 Fall 2021

Section 18: Sine and Cosine Series

This section has two topics

- Fourier series of functions on $(-p, p)$ that have symmetry, and
- half-range sine and cosine series for functions defined on $(0, p)$.

Functions with Symmetry

Recall some definitions:

Suppose f is defined on an interval containing x and $-x$.

If $f(-x) = f(x)$ for all x, then f is said to be **even.**

If $f(-x) = -f(x)$ for all x, then f is said to be **odd.**

For example, $f(x) = x^n$ is even if n is even and is odd if n is odd. The trigonometric function $g(x) = \cos x$ is even, and $h(x) = \sin x$ is odd.

Even and Odd Symmetry

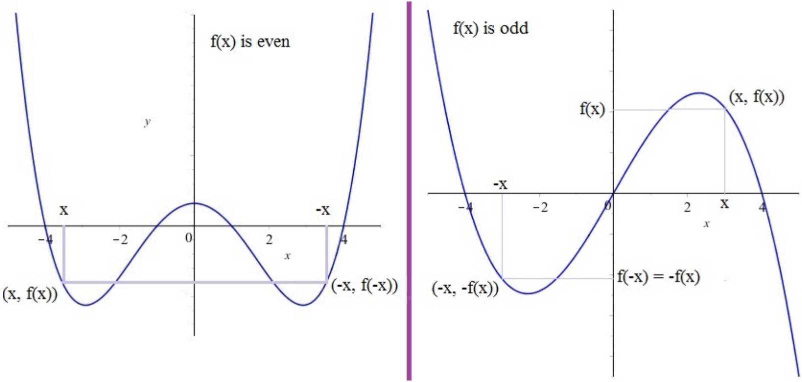


Figure: Graphical interpretation of even and odd symmetry.

Integrals on symmetric intervals

If f is an even function on $(-p, p)$, then

$$\int_{-p}^{p} f(x)\,dx = 2\int_{0}^{p} f(x)\,dx.$$

If f is an odd function on $(-p, p)$, then

$$\int_{-p}^{p} f(x)\,dx = 0.$$

Products of Even and Odd functions

$$\text{Even} \quad \times \quad \text{Even} \quad = \quad \text{Even},$$

and

$$\text{Odd} \quad \times \quad \text{Odd} \quad = \quad \text{Even}.$$

While

$$\text{Even} \quad \times \quad \text{Odd} \quad = \quad \text{Odd}.$$

So, suppose f **is even** on $(-p,p)$**.** This tells us that $f(x)\cos(nx)$ is even for all n and $f(x)\sin(nx)$ is odd for all n**.**

And, if f **is odd** on $(-p,p)$**.** This tells us that $f(x)\sin(nx)$ is even for all n and $f(x)\cos(nx)$ is odd for all n

Fourier Series of an Even Function

If f is even on $(-p, p)$, then the Fourier series of f has only constant and cosine terms. Moreover

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{p}\right)$$

where

$b_n = 0$
for all n

$$a_0 = \frac{2}{p}\int_0^p f(x)\,dx$$

and

$$a_n = \frac{2}{p}\int_0^p f(x)\cos\left(\frac{n\pi x}{p}\right)\,dx.$$

Fourier Series of an Odd Function

If f is odd on $(-p, p)$, then the Fourier series of f has only sine terms. Moreover

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{p}\right)$$

where

$$b_n = \frac{2}{p}\int_0^p f(x)\sin\left(\frac{n\pi x}{p}\right)\,dx.$$

$a_0 = 0$

$a_n = 0$ for all n

Find the Fourier series of f

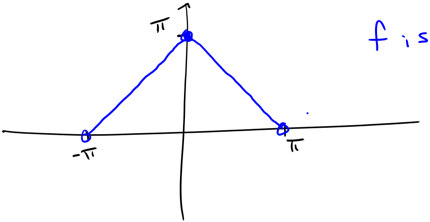
$$f(x) = \begin{cases} x+\pi, & -\pi < x < 0 \\ \pi - x, & 0 \le x < \pi \end{cases}$$

To determine symmetry, try evaluating $f(-x)$.

Plotting is another option

$y = x+\pi$

$y = \pi - x$



f is even

$b_n = 0$ for all n

Here $p = \pi$

$$\frac{n\pi x}{p} = \frac{n\pi x}{\pi} = nx$$

Let's compute the a's.

Using symmetry

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} (\pi - x)\,dx$$

$$= \frac{2}{\pi}\left[\pi x - \frac{x^2}{2}\right|_0^{\pi} = \frac{2}{\pi}\left[\pi^2 - \frac{\pi^2}{2} - 0\right]$$

$$= \frac{2}{\pi}\left(\frac{\pi^2}{2}\right) = \pi \qquad a_0 = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} (\pi - x)\cos(nx)\,dx$$

Int by parts $\quad u = \pi - x, \quad du = -dx$

$$dv = \cos(nx)\,dx \qquad v = \frac{1}{n}\sin(nx)$$

$$a_n = \frac{2}{\pi}\left[\underbrace{\frac{\pi - x}{n}\sin(nx)\Big|_0^{\pi}}_{0} + \int_0^{\pi} \frac{1}{n}\sin(nx)\,dx\right.$$

$$= \frac{2}{\pi}\left[\frac{-1}{n^2}\cos(nx)\right|_0^{\pi}$$

$$= \frac{-2}{n^2\pi}\left[\cos(n\pi) - \cos(0)\right]$$

$$= \frac{-2}{n^2\pi}\left((-1)^n - 1\right)$$

$$a_n = \frac{2}{n^2\pi}\left(1 - (-1)^n\right)$$

$$a_0 = \pi$$

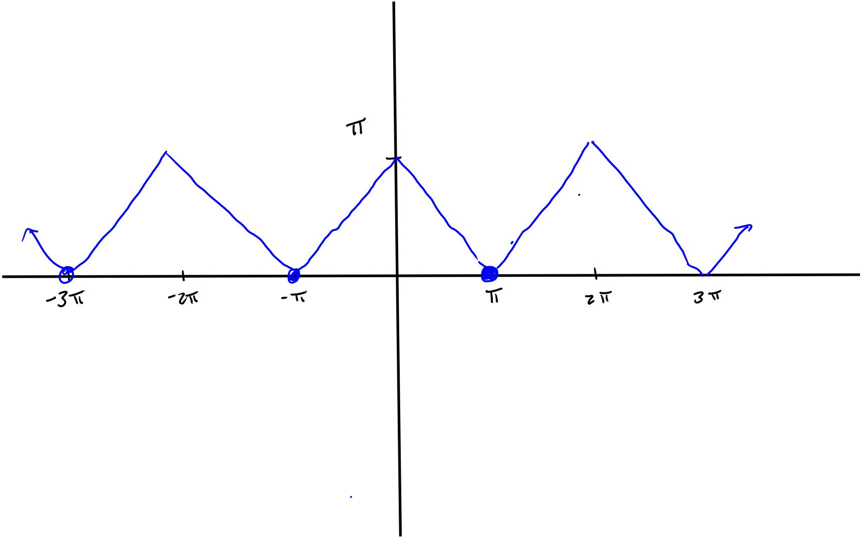
Hence

$$\boxed{f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi}\left(1-(-1)^n\right)\cos(nx)}$$

for $f(x) = \begin{cases} \pi + x, & -\pi < x < 0 \\ \pi - x, & 0 \le x < \pi \end{cases}$

Let's plot what the series converges to

on $(-3\pi, 3\pi)$


π
-3π
-2π
-π
π
2π
3π

Taking Advantage of Symmetry

$$f(x) = \begin{cases} x+\pi, & -\pi < x < 0 \\ \pi - x, & 0 \le x < \pi \end{cases}$$

Notice the amount of work we saved by using the symmetry. The formulas for the coefficients give

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{0}(x+\pi)\,dx + \frac{1}{\pi}\int_0^{\pi}(\pi - x)\,dx \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{0}(x+\pi)\cos(nx)\,dx + \frac{1}{\pi}\int_0^{\pi}(\pi - x)\cos(nx)\,dx \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{0}(x+\pi)\sin(nx)\,dx + \frac{1}{\pi}\int_0^{\pi}(\pi - x)\sin(nx)\,dx \end{aligned}$$

We only had to compute the integrals in **green**.

Half Range Sine and Half Range Cosine Series

Suppose f is only defined for $0 < x < p$. We can **extend** f to the left, to the interval $(-p, 0)$, as either an even function or as an odd function. Then we can express f with **two distinct** series.

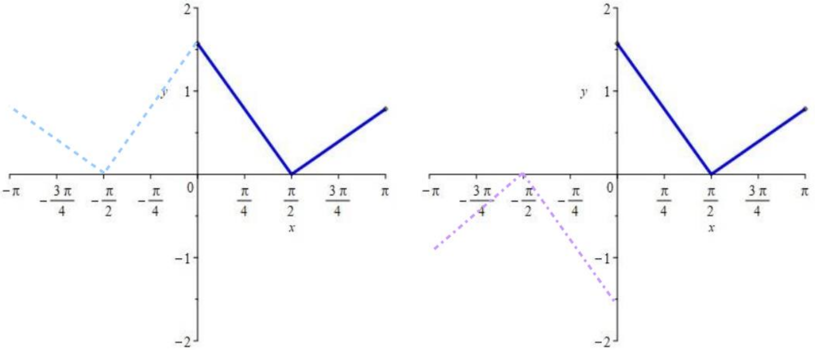


Figure: Some function f shown in dark blue with even and odd extensions.

Half Range Cosine Series

For f defined on $(0,p)$ we can *pretend* that f is defined on $(-p,p)$ by setting

$$f(-x)=f(x)\quad 0<x<p.$$

Then we can define the

Half range cosine series $$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi x}{p}\right)$$

where $$a_0=\frac{2}{p}\int_0^p f(x)\,dx \quad \text{and} \quad a_n=\frac{2}{p}\int_0^p f(x)\cos\left(\frac{n\pi x}{p}\right)\,dx.$$

Extending a Function to be Even

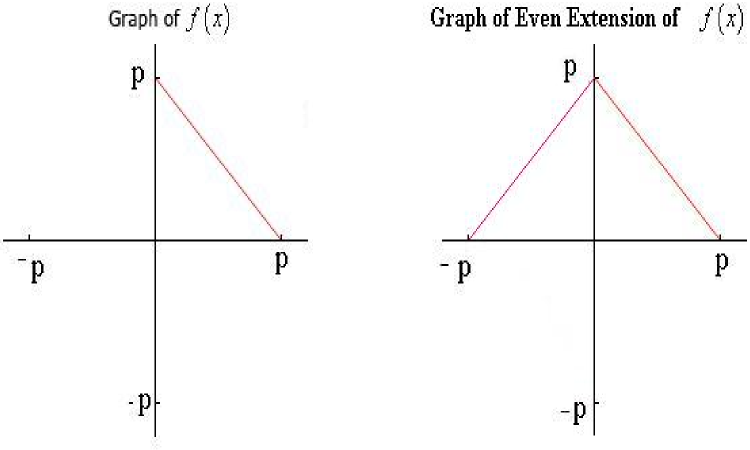


Figure: $f(x) = p - x$, $0 < x < p$ together with its **even** extension.

Half Range Sine

Suppose f is defined on $(0, p)$. We can extend f to be defined on $(-p, p)$ by setting

$$f(-x) = -f(x) \quad 0 < x < p.$$

Then we can define the

$$\textbf{Half range sine series} \quad f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{p}\right)$$

$$\text{where} \quad b_n = \frac{2}{p}\int_0^p f(x)\sin\left(\frac{n\pi x}{p}\right)\,dx.$$

Extending a Function to be Odd

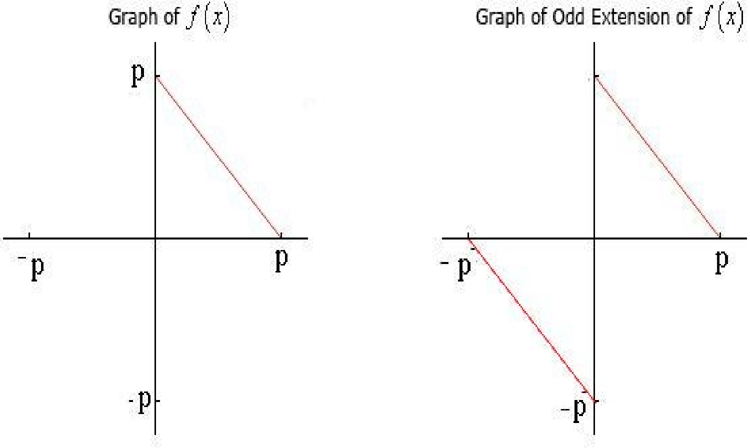


Figure: $f(x) = p - x$, $0 < x < p$ together with its **odd** extension.

Find the Half Range Sine Series of f

$$f(x) = 2 - x, \quad 0 < x < 2$$

$p = 2$, so $\dfrac{n\pi x}{p} = \dfrac{n\pi x}{2}$

$2/p$

$$b_n = \frac{2}{2}\int_0^2 f(x)\sin\left(\frac{n\pi x}{2}\right)dx$$

$$= \int_0^2 (2-x)\sin\left(\frac{n\pi x}{2}\right)dx$$

Int by parts, $u = 2-x$, $du = -dx$

$dv = \sin\left(\frac{n\pi x}{2}\right)dx$ $\quad v = \frac{-2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)$