November 17 Math 2306 sec. 52 Spring 2023

Section 16: Laplace Transforms of Derivatives and IVPs

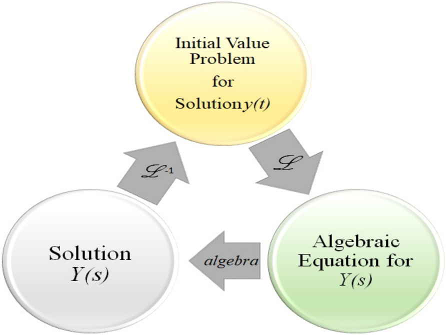


Figure: We'll use the Laplace transform as a tool for solving certain IVPs and systems of IVPs. Our use will be restricted to IVPs with **constant coefficients** and initial conditions given at $t = 0$.

The Laplace Transform of Derivatives

For $y = y(t)$ defined on $[0, \infty)$ having derivatives y', y'' and so forth, if $\mathscr{L}\{y(t)\} = Y(s)$, then

$$\begin{aligned}
\mathscr{L}\left\{\frac{dy}{dt}\right\} &= sY(s) - y(0) \\
\mathscr{L}\left\{\frac{d^2y}{dt^2}\right\} &= s^2Y(s) - sy(0) - y'(0) \\
\mathscr{L}\left\{\frac{d^3y}{dt^3}\right\} &= s^3Y(s) - s^2y(0) - sy'(0) - y''(0) \\
&\vdots \\
\mathscr{L}\left\{\frac{d^ny}{dt^n}\right\} &= s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - y^{(n-1)}(0).
\end{aligned}$$

Solving and IVP

Use the Laplace transform to solve the initial value problem.

$$y'' - 6y' + 8y = t, \quad y(0) = 1, \quad y'(0) = 2$$

We set $Y(s) = \mathscr{L}\{y(t)\}$ and took the Laplace transform of both sides of the ODE. Using the results for the derivatives, this gives

$$s^2 Y(s) - sy(0) - y'(0) + 6\,(sY(s) - y(0)) + 8\,Y(s) = \frac{1}{s^2}.$$

We substituted in the initial conditions, $y(0) = 1$ and $y'(0) = 2$, and used algebra to isolate $Y(s)$. We had

$$(s^2 - 6s + 8)Y(s) = s - 4 + \frac{1}{s^2}.$$

Solving and IVP

Use the Laplace transform to solve the initial value problem.

$$y'' - 6y' + 8y = t, \quad y(0) = 1, \quad y'(0) = 2$$

$$\boxed{(s^2 - 6s + 8)Y(s) = s - 4 + \frac{1}{s^2}}$$

The coefficient of Y at this step is the **characteristic polynomial** of the original ODE. This will **ALWAYS** happen. What we end up with is

$$Y(s) = \frac{s-4}{s^2 - 6s + 8} + \frac{\frac{1}{s^2}}{s^2 - 6s + 8}.$$

To finish the solution process, we have to take the inverse transform

$$y(t) = \mathscr{L}^{-1}\{Y(s)\}.$$

$$Y(s) = \frac{s-4}{s^2-6s+8} + \frac{\frac{1}{s^2}}{s^2-6s+8}.$$

$s^2 - 6s + 8 = (s-2)(s-4)$

$$Y(s) = \frac{s-4}{(s-2)(s-4)} + \frac{1}{s^2(s-2)(s-4)}$$

$$= \frac{1}{s-2} + \frac{1}{s^2(s-2)(s-4)}$$

Partial fractions

$$\frac{1}{s^2(s-2)(s-4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-2} + \frac{D}{s-4}$$

$A = \frac{3}{32}$, $B = \frac{1}{8}$, $C = \frac{-1}{8}$, $D = \frac{1}{32}$

$$Y(s) = \frac{1}{s-2} + \frac{\frac{3}{32}}{s} + \frac{\frac{1}{8}}{s^2} - \frac{\frac{1}{8}}{s-2} + \frac{\frac{1}{32}}{s-4}$$

$$Y(s) = \frac{3}{32}\left(\frac{1}{s}\right) + \frac{1}{8}\left(\frac{1!}{s^2}\right) + \frac{7}{8}\left(\frac{1}{s-2}\right) + \frac{1}{32}\left(\frac{1}{s-4}\right)$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

The solution to the IVP is

$$y(t) = \frac{3}{32} + \frac{1}{8}t + \frac{7}{8}e^{2t} + \frac{1}{32}e^{4t}$$

Does $y(0)=1$?

$$\frac{3}{32} + \frac{7}{8} + \frac{1}{32} = \frac{4}{32} + \frac{7}{8} = \frac{1}{8} + \frac{7}{8} = 1$$

Use Laplace Transforms to Solve and IVP

- Start with constant coefficient IVP with IC at $t = 0$. For example[a]

$$ay'' + by' + cy = g(t), \quad y(0) = y_0, \quad y'(0) = y_1.$$

- Let $Y(s) = \mathscr{L}\{y(t)\}$ and take the transform of both sides of the ODE using any necessary results.
- Sub in the initial conditions where they appear in the transformed equation.
- Use basic algebra to isolate the transform $Y(s)$.
- Using whatever algebra or function identities that are needed, take the inverse transform to obtain the solution

$$y(t) = \mathscr{L}^{-1}\{Y(s)\}.$$

[a] The IVP can be of any order.

Input & State Responses

Note that our solution has the basic format

$$Y(s) = \frac{Q(s)}{P(s)} + \frac{G(s)}{P(s)}$$

where Q is a polynomial with coefficients determined by the initial conditions, G is the Laplace transform of $g(t)$ and P is the **characteristic polynomial** of the original equation. The solution y consists of two corresponding terms.

Zero Input Response

$$\mathscr{L}^{-1}\left\{\frac{Q(s)}{P(s)}\right\}$$ is called the **zero input response**,

and

Zero State Response

$$\mathscr{L}^{-1}\left\{\frac{G(s)}{P(s)}\right\}$$ is called the **zero state response**.

Input & State Responses

The zero input and zero state responses are related to the complementary and particular solutions, but they are not quite the same since they are related to initial value problems as opposed to simply the differential equation.

Zero Input Response

The **zero input response** satisfies the initial value problem with homogeneous differential equation and nonhomogeneous initial conditions,

$$ay'' + by' + cy = 0, \quad y(0) = y_0, \quad y'(0) = y_1.$$

Zero State Response

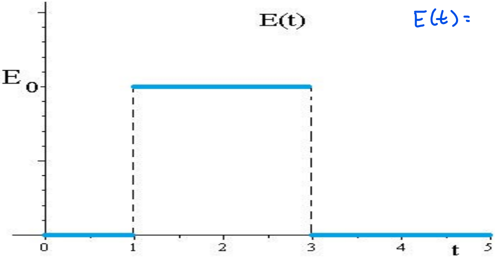
The **zero state response** satisfies the initial value problem with nonhomogeneous differential equation and homogeneous initial conditions,

$$ay'' + by' + cy = g(t), \quad y(0) = 0, \quad y'(0) = 0.$$

An IVP with Piecewise Input

One of the most powerful uses of the Laplace transform is in applications that involve a piecewise defined forcing function. Let's look at an example.

An LR-series circuit has inductance $L = 1$h, resistance $R = 10\Omega$, and implied voltage $E(t)$ whose graph is given below. If the initial current $i(0) = 0$, find the current $i(t)$ in the circuit.



$$E(t) = \begin{cases} 0, & 0 \le t < 1 \\ E_0, & 1 \le t < 3 \\ 0, & t \ge 3 \end{cases}$$

Figure: A switch is closed for two seconds from $t = 1$ until $t = 3$ during which a constant E_0 volts is applied.

$E(t) = 0 - 0\,u(t-1) + E_0\,u(t-1) - E_0\,u(t-3) + 0\,u(t-3)$

An IVP with Piecewise Input

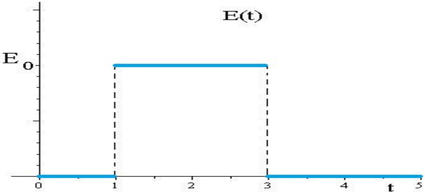


Figure: $L\dfrac{di}{dt} + Ri = E(t)$, $i(0) = 0$ where $L = 1$, $R = 10$ and $E(t)$ is shown.

$1\dfrac{di}{dt} + 10i = E_0\,u(t-1) - E_0\,u(t-3) \qquad i(0) = 0.$

Let $I(s) = \mathcal{L}\{i(t)\}$.

$\mathcal{L}\{i' + 10i\} = \mathcal{L}\{E_0\,u(t-1) - E_0\,u(t-3)\}$

$\mathcal{L}\{i'\} + 10\,\mathcal{L}\{i\} = E_0\,\mathcal{L}\{u(t-1)\} - E_0\,\mathcal{L}\{u(t-3)\}$

$$sI(s) - \underset{0}{i(0)} + 10\,I(s) = E_0\frac{e^{-1s}}{s} - E_0\frac{e^{-3s}}{s}$$

$$(s+10)\,I(s) = E_0\frac{e^{-s}}{s} - E_0\frac{e^{-3s}}{s}$$

$$I(s) = E_0\frac{e^{-s}}{s(s+10)} - E_0\frac{e^{-3s}}{s(s+10)}$$

$$\frac{E_0 e^{-s}}{s(s+10)} = e^{-s}\underbrace{\frac{E_0}{s(s+10)}}_{F(s)}$$

Let $f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{E_0}{s(s+10)}\right\}$

$$\frac{E_0}{s(s+10)} = \frac{A}{s} + \frac{B}{s+10} \Rightarrow \frac{\frac{E_0}{10}}{s} - \frac{\frac{E_0}{10}}{s+10}$$

$$f(t) = \mathcal{L}^{-1}\left\{ \frac{\frac{E_0}{10}}{s} - \frac{\frac{E_0}{10}}{s+10} \right\}$$

$$= \frac{E_0}{10}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{E_0}{10}\mathcal{L}^{-1}\left\{\frac{1}{s+10}\right\}$$

$$f(t) = \frac{E_0}{10} - \frac{E_0}{10}e^{-10t}$$

$$I(s) = \frac{E_0}{s(s+10)}e^{-s} - \frac{E_0}{s(s+10)}e^{-3s}$$

$$\mathcal{L}^{-1}\left\{e^{-as}F(s)\right\} = f(t-a)\mathcal{U}(t-a)$$

The current $i(t) = \mathcal{L}^{-1}\{I(s)\}$

$$i(t) = \left(\frac{E_0}{10} - \frac{E_0}{10}e^{-10(t-1)}\right)\mathcal{U}(t-1) - \left(\frac{E_0}{10} - \frac{E_0}{10}e^{-10(t-3)}\right)\mathcal{U}(t-3)$$

Let's identify $i(t)$ in the traditional stacked notation.

For $0 \le t < 1$, $u(t-1) = 0$ $u(t-3) = 0$

$$i(t) = 0$$

For $1 \le t < 3$, $u(t-1) = 1$ $u(t-3) = 0$

$$i(t) = \frac{E_0}{10} - \frac{E_0}{10} e^{-10(t-1)}$$

For $t \geq 3$, $u(t-1) = 1$ and $u(t-3) = 1$

$$i(t) = \frac{E_0}{10} - \frac{E_0}{10} e^{-10(t-1)} - \left(\frac{E_0}{10} - \frac{E_0}{10} e^{-10(t-3)} \right)$$

$$i(t) = \frac{E_0}{10} e^{-10(t-3)} - \frac{E_0}{10} e^{-10(t-1)}$$

The current

$$i(t) = \begin{cases} 0, & 0 \le t < 1 \\ \frac{E_0}{10} - \frac{E_0}{10} e^{-10(t-1)}, & 1 \le t < 3 \\ \frac{E_0}{10} e^{-10(t-3)} - \frac{E_0}{10} e^{-10(t-1)}, & t \ge 3 \end{cases}$$