November 17 Math 2306 sec. 54 Fall 2021

Section 18: Sine and Cosine Series

This section has two topics

- Fourier series of functions on $(-p, p)$ that have symmetry, and
- half-range sine and cosine series for functions defined on $(0, p)$.

Functions with Symmetry

Recall some definitions:

Suppose f is defined on an interval containing x and $-x$.

If $f(-x) = f(x)$ for all x, then f is said to be **even.**

If $f(-x) = -f(x)$ for all x, then f is said to be **odd.**

For example, $f(x) = x^n$ is even if n is even and is odd if n is odd. The trigonometric function $g(x) = \cos x$ is even, and $h(x) = \sin x$ is odd.

Even and Odd Symmetry

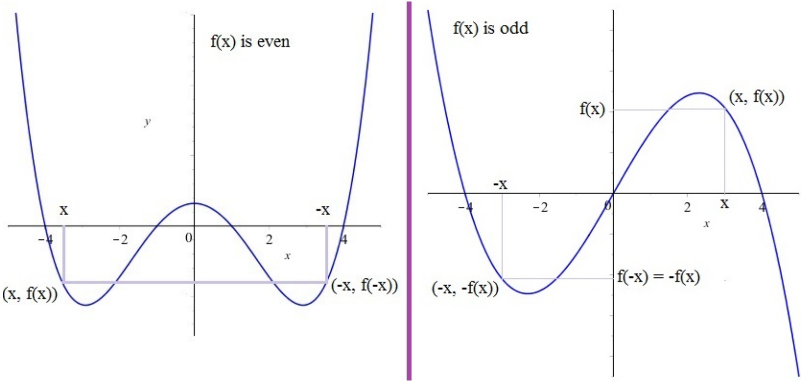


Figure: Graphical interpretation of even and odd symmetry.

Integrals on symmetric intervals

If f is an even function on $(-p, p)$, then

$$\int_{-p}^{p} f(x)\,dx = 2\int_{0}^{p} f(x)\,dx.$$

If f is an odd function on $(-p, p)$, then

$$\int_{-p}^{p} f(x)\,dx = 0.$$

Products of Even and Odd functions

$$\text{Even} \quad \times \quad \text{Even} \quad = \quad \text{Even},$$

and

$$\text{Odd} \quad \times \quad \text{Odd} \quad = \quad \text{Even}.$$

While

$$\text{Even} \quad \times \quad \text{Odd} \quad = \quad \text{Odd}.$$

So, suppose f **is even** on $(-p, p)$. This tells us that $f(x)\cos(nx)$ is even for all n and $f(x)\sin(nx)$ is odd for all n.

And, if f **is odd** on $(-p, p)$. This tells us that $f(x)\sin(nx)$ is even for all n and $f(x)\cos(nx)$ is odd for all n

Fourier Series of an Even Function

If f is even on $(-p, p)$, then the Fourier series of f has only constant and cosine terms. Moreover

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{p}\right)$$

where

$b_n = 0$ for all n

$$a_0 = \frac{2}{p}\int_0^p f(x)\,dx$$

and

$$a_n = \frac{2}{p}\int_0^p f(x)\cos\left(\frac{n\pi x}{p}\right)\,dx.$$

Fourier Series of an Odd Function

If f is odd on $(-p, p)$, then the Fourier series of f has only sine terms. Moreover

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{p}\right)$$

where

$$b_n = \frac{2}{p}\int_0^p f(x)\sin\left(\frac{n\pi x}{p}\right)\,dx.$$

$a_0 = 0$

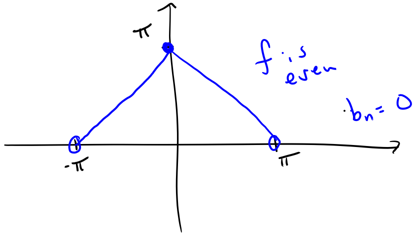
$a_n = 0$ for all $n \geq 1$

Find the Fourier series of f

$$f(x) = \begin{cases} x+\pi, & -\pi < x < 0 \\ \pi - x, & 0 \le x < \pi \end{cases}$$

$y = x+\pi$
slope 1, y-int. π

$y = \pi - x$
slope -1



f is even

$b_n = 0$

Let's find the a's.

Symmetry can be determined by evaluating $f(-x)$. or by looking at the graph.

$P = \pi$, $\dfrac{n\pi x}{P} = \dfrac{n\pi x}{\pi} = nx$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} (\pi - x)\,dx$$

$$= \frac{2}{\pi}\left[\pi x - \frac{x^2}{2}\right]_0^{\pi} = \frac{2}{\pi}\left[\pi^2 - \frac{\pi^2}{2}\right] = \frac{2}{\pi}\left[\frac{\pi^2}{2}\right]$$

$$= \pi \qquad \boxed{a_0 = \pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} (\pi - x)\cos(nx)\,dx$$

Int by parts $\quad u = \pi - x, \quad du = -dx$

$$dv = \cos(nx)dx \qquad v = \frac{1}{n}\sin(nx)$$

$$a_n = \frac{2}{\pi}\left[\frac{\pi - x}{n} \cancel{\sin}(nx) \Big|_0^{\pi} + \int_0^{\pi} \frac{1}{n}\sin(nx)\,dx \right.$$

$$0$$

$$= \frac{2}{\pi}\left[\frac{-1}{n^2}\cos(nx) \Big|_0^{\pi} \right.$$

$$= \frac{-2}{n^2\pi}\left[\cos(n\pi) - \cos(0) \right]$$

$$= \frac{-2}{n^2\pi}\left((-1)^n - 1 \right)$$

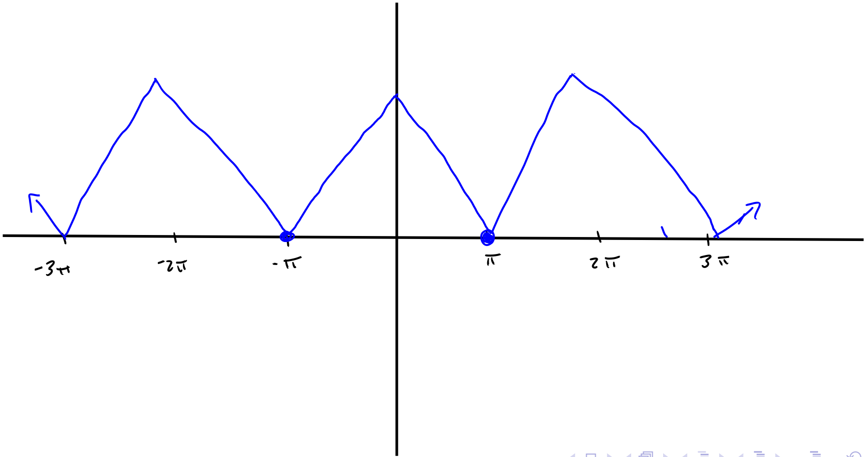
$$a_n = \frac{2}{n^2\pi}\left(1 - (-1)^n\right) \qquad a_0 = \pi$$

$$\boxed{\text{Hence} \quad f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2 \pi} \left(1 - (-1)^n\right) \cos(nx)}$$

for

$$f(x) = \begin{cases} x + \pi, & -\pi < x < 0 \\ \pi - x, & 0 \le x < \pi \end{cases}$$

Let's plot what the series converges to over the interval $-3\pi < x < 3\pi$



Taking Advantage of Symmetry

$$f(x) = \begin{cases} x+\pi, & -\pi < x < 0 \\ \pi - x, & 0 \le x < \pi \end{cases}$$

Notice the amount of work we saved by using the symmetry. The formulas for the coefficients give

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{0}(x+\pi)\,dx + \frac{1}{\pi}\int_0^{\pi}(\pi - x)\,dx \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{0}(x+\pi)\cos(nx)\,dx + \frac{1}{\pi}\int_0^{\pi}(\pi - x)\cos(nx)\,dx \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{0}(x+\pi)\sin(nx)\,dx + \frac{1}{\pi}\int_0^{\pi}(\pi - x)\sin(nx)\,dx \end{aligned}$$

We only had to compute the integrals in **green**.

Half Range Sine and Half Range Cosine Series

Suppose f is only defined for $0 < x < p$. We can **extend** f to the left, to the interval $(-p, 0)$, as either an even function or as an odd function. Then we can express f with **two distinct** series.

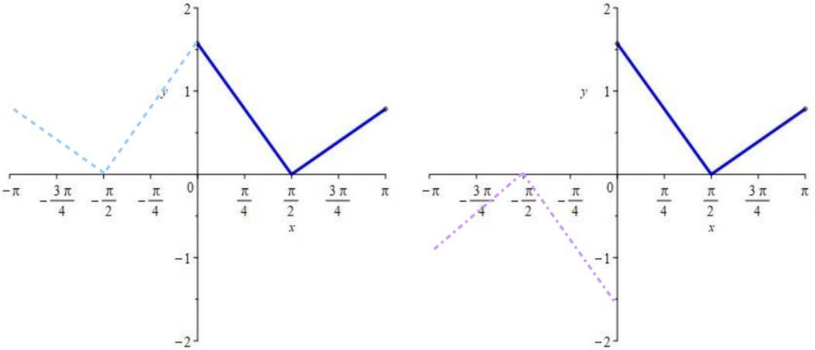


Figure: Some function f shown in dark blue with even and odd extensions.

Half Range Cosine Series

For f defined on $(0, p)$ we can *pretend* that f is defined on $(-p, p)$ by setting

$$f(-x) = f(x) \quad 0 < x < p.$$

Then we can define the

Half range cosine series $\quad f(x) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \cos\left(\dfrac{n\pi x}{p}\right)$

where $\quad a_0 = \dfrac{2}{p}\displaystyle\int_0^p f(x)\,dx \quad$ and $\quad a_n = \dfrac{2}{p}\displaystyle\int_0^p f(x)\cos\left(\dfrac{n\pi x}{p}\right)\,dx.$

Extending a Function to be Even

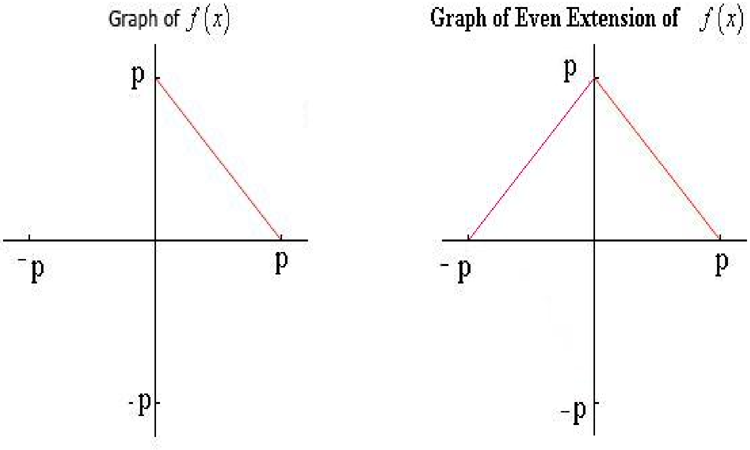


Figure: $f(x) = p - x$, $0 < x < p$ together with its **even** extension.

Half Range Sine

Suppose f is defined on $(0, p)$. We can extend f to be defined on $(-p, p)$ by setting

$$f(-x) = -f(x) \quad 0 < x < p.$$

Then we can define the

Half range sine series $\quad f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{p}\right)$

where $\quad b_n = \frac{2}{p}\int_0^p f(x)\sin\left(\frac{n\pi x}{p}\right)\,dx.$

Extending a Function to be Odd

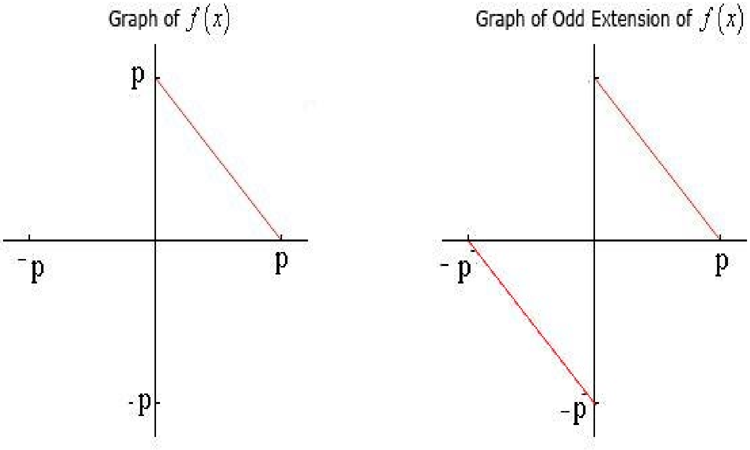


Figure: $f(x) = p - x$, $0 < x < p$ together with its **odd** extension.

Find the Half Range Sine Series of f

$$f(x) = 2 - x, \quad 0 < x < 2$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{p}\right), \quad b_n = \frac{2}{p}\int_0^p f(x) \sin\left(\frac{n\pi x}{p}\right) dx$$

$$p = 2, \quad \frac{n\pi x}{p} = \frac{n\pi x}{2}$$

$2/p$

$$b_n = \frac{2}{2}\int_0^2 (2-x)\sin\left(\frac{n\pi x}{2}\right) dx$$

Int by parts: $u = 2 - x, \quad du = -dx$

$$dv = \sin\left(\frac{n\pi x}{2}\right) dx \qquad v = \frac{-2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)$$

$$b_n = \frac{-2}{n\pi}(2-x)\cos\left(\frac{n\pi x}{2}\right)\Bigg|_0^2 - \cancelto{0}{\int_0^2 \frac{-2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)(-1)\,dx}$$

$$= \frac{-2}{n\pi}(\cancelto{0}{2-2})\cos(n\pi) - \frac{-2}{n\pi}(2-0)\cos(0)$$

$$b_n = \frac{4}{n\pi}$$

The half range sine series is

$$f(x) = \sum_{n=1}^{\infty} \frac{4}{n\pi}\sin\left(\frac{n\pi x}{2}\right)$$