November 17 Math 3260 sec. 51 Fall 2025

5.3 Visualizing Linear Transformations

We want to consider certain linear mappings from R^2 to R^2 that correspond to geometric transformations.

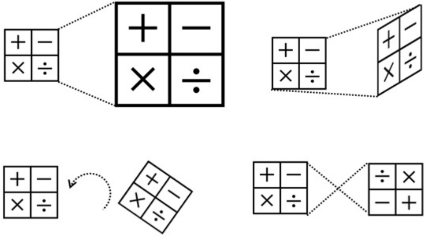


Figure: Scaling, shearing, rotations, reflections

Notational Note: The notation $\vec{x} \mapsto \vec{y}$ is read "$\vec{x}$ maps to $\vec{y}$." It's a common short hand for $T(\vec{x}) = \vec{y}$ when T is some transformation.

Scaling, Shear, and Rotation

- $T : R^2 \to R^2$ given by $T(\vec{x}) = r\vec{x}$ is a dilation if $r > 1$ and a contraction if $0 < r < 1$. The standard matrix

$$A = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} = rI_2.$$

- $S : R^2 \to R^2$ given by $\vec{x} \mapsto A\vec{x}$ is a shear transformation if

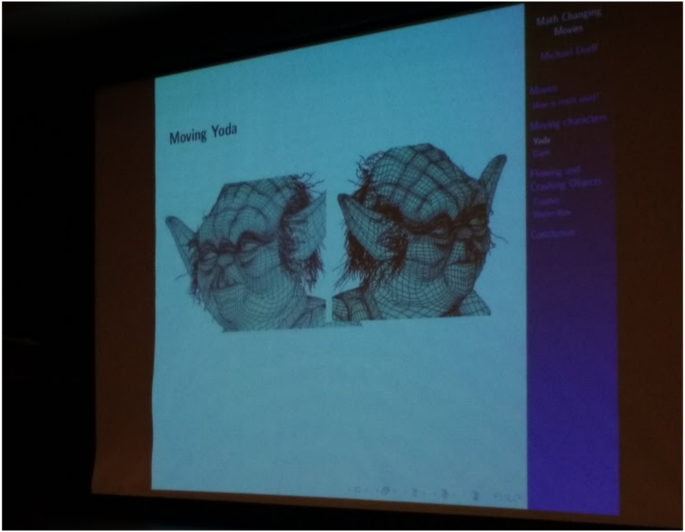
$$A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}, \quad \text{or} \quad A = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}, \quad k \in R.$$

- $R_\theta : R^2 \to R^2$ given by $\vec{x} \mapsto A_\theta\vec{x}$ is a rotation about the origin by an angle θ in the c.clockwise direction, where

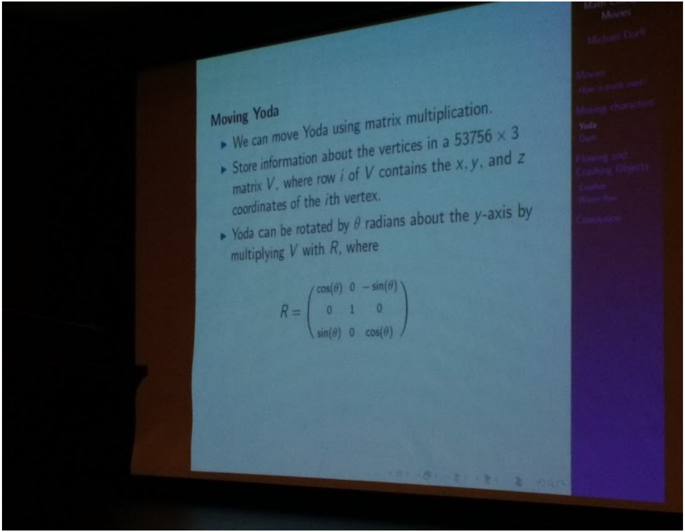
$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$R_{-\theta} = R_\theta^{-1}$

Rotation in Animation



Rotation in Animation



Reflection Through Axis

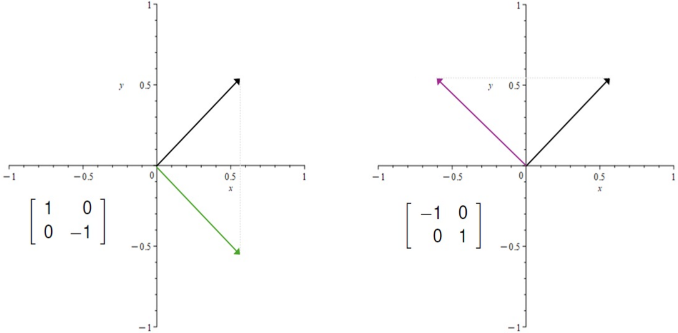


Figure: The matrix to reflect through the x_1-axis (left) or x_2-axis (right).

$$P_{x_1} : R^2 \to R^2 \quad P_{x_1}(\langle x_1, x_2 \rangle) = \langle x_1, -x_2 \rangle$$

$$P_{x_2} : R^2 \to R^2 \quad P_{x_2}(\langle x_1, x_2 \rangle) = \langle -x_1, x_2 \rangle$$

Summary of Geometric Transformations on R^2

- **Scaling**: $\vec{x} \mapsto rI_2\vec{x}$, is a dilation if $r > 1$ and contraction if $0 < r < 1$.
- **Shear**: $\vec{x} \mapsto A\vec{x}$ where $A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ or $A = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$ for constant k.
- **Rotation** (counter clockwise about the origin through angle θ): $\vec{x} \mapsto A_\theta\vec{x}$ where $A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$
- **Reflection** (through an axis):
  $P_{x_1}(\langle x_1, x_2\rangle) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \langle x_1, x_2\rangle = \langle x_1, -x_2\rangle$, or
  $P_{x_2}(\langle x_1, x_2\rangle) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \langle x_1, x_2\rangle = \langle -x_1, x_2\rangle$.

We can combine these with the composition of transformations.

5.5 Compositions & Similarity

Suppose $S : R^n \to R^p$ and $T : R^p \to R^m$ are linear transformations, then we can ask about the composition

$$T \circ S : R^n \to R^m.$$

$$T(S(\vec{x})) = T(A_S\vec{x}) = A_T(A_S\vec{x}) = (A_TA_S)\vec{x}$$

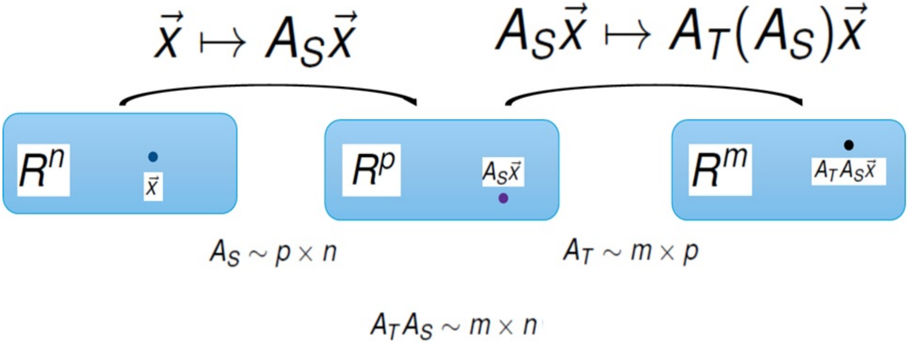
Suppose

$$S(\vec{x}) = A_S\vec{x}, \quad \text{and} \quad T(\vec{y}) = A_T\vec{y}.$$

How is the standard matrix for the composition related to the standard matrices of S and T?

This give the primary motivation for the way matrix multiplication is defined.

Matrix Multiplication is Composition



$m \times p$ $p \times n$

Figure: $\vec{x}$ is mapped from R^n to R^p, then $A_S\vec{x}$ is mapped from R^p to R^m. The composition maps from R^n to R^m.

$$
\begin{aligned}
S : R^n \longrightarrow R^p \quad &\Longrightarrow \quad A_S \sim p \times n \\
T : R^p \longrightarrow R^m \quad &\Longrightarrow \quad A_T \sim m \times p \\
T \circ S : R^n \longrightarrow R^m \quad &\Longrightarrow \quad A_TA_S \sim m \times n
\end{aligned}
$$

Example

Suppose that $S : R^3 \to R^2$ is the linear transformation

$$S(\langle x_1, x_2, x_3 \rangle) = \langle 2x_1 + x_2, 2x_1 + x_2 + x_3 \rangle$$

and suppose that $T : R^2 \to R^3$ is the linear transformation

$$T(\langle x_1, x_2 \rangle) = \langle -x_1, 3x_1 - x_2, -2x_1 + 3x_2 \rangle .$$

Find the standard matrix for the composition $T \circ S$.

For A_S, we need $S(\vec{e}_1)$, $S(\vec{e}_2)$, $S(\vec{e}_3)$

$S(\langle 1,0,0 \rangle) = \langle 2,2 \rangle$, $S(\langle 0,1,0 \rangle) = \langle 1,1 \rangle$

$S(\langle 0,0,1 \rangle) = \langle 0,1 \rangle$ $\quad A_S = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}.$

For A_T, we need $T(\vec{e}_1)$ and $T(\vec{e}_2)$

$T(\langle 1,0 \rangle) = \langle -1, 3, -2 \rangle$, $T(\langle 0,1 \rangle) = \langle 0, -1, 3 \rangle$

$$A_T = \begin{bmatrix} -1 & 0 \\ 3 & -1 \\ -2 & 3 \end{bmatrix}$$

$$A_{T\circ S} = A_T A_S = \underbrace{\begin{bmatrix} -1 & 0 \\ 3 & -1 \\ -2 & 3 \end{bmatrix}}_{3\times 2} \underbrace{\begin{bmatrix} 2 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}}_{2\times 3}$$

$$= \begin{bmatrix} -2 & -1 & 0 \\ 4 & 2 & -1 \\ 2 & 1 & 3 \end{bmatrix}$$

$$(T\circ S)(\vec{e}_1) = \langle -2, 4, 2 \rangle$$

Reflection in Line Through the Origin

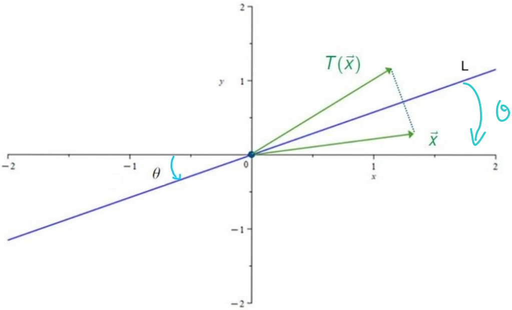


Figure: We want a transformation T to reflect a vector through a line through the origin that makes an angle θ with the x_1-axis.

We'll do this in three steps.

Start w/ Line & Vector

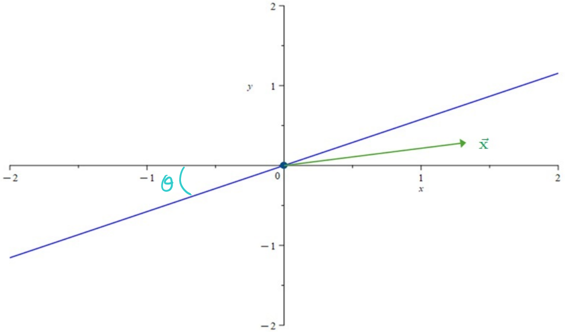


Figure: The line L makes an angle θ with respect to the x_1-axis. We want to reflect the vector $\vec{x}$ through it.

$$\text{Apply:}\quad R_{-\theta}(\vec{x}) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\vec{x}, \quad \text{let} \quad \vec{y} = R_{-\theta}(\vec{x})$$

Rotate θ Clockwise

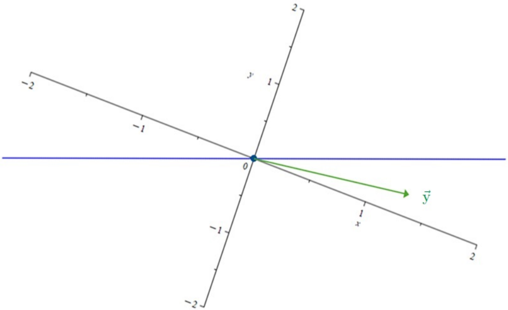


Figure: Rotate through θ clockwise using $R_{-\theta}$. L becomes the x_1-axis.

$$\text{Next apply:}\quad P_{x_1}(\vec{y}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \vec{y}, \quad \text{let} \quad \vec{z} = P_{x_1}(\vec{y})$$

Reflect Through x_1-axis

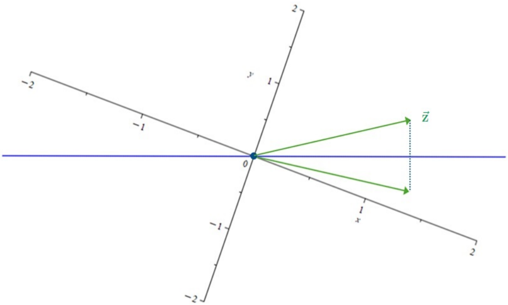


Figure: Reflect through the x_1-axis using the Reflection transformation P_{x_1}.

Finally apply: $R_\theta(\vec{z}) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \vec{z}$, this is $T(\vec{x})$, $T(\vec{x}) = R_\theta(\vec{z})$

Rotate θ Counter Clockwise

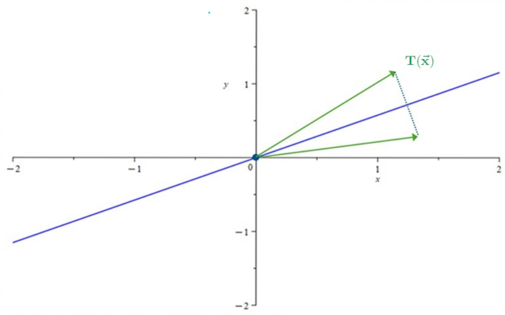


Figure: Then we rotation back through θ in the counterclockwise direction by applying the transformation R_θ.

So $T(\vec{x}) = R_\theta(P_{x_1}(R_{-\theta}(\vec{x})))$, and the transformation is the composition

$$T = R_\theta \circ P_{x_1} \circ R_{-\theta}$$

inverses

Similarity

Our complicated reflection through a line that was not horizontal can be done with the “simple” reflection through a horizontal line. Note that the matrix for this is the product

$$A_T = A_{-\theta}^{-1} A_{P_{x_1}} A_{-\theta}.$$

A_θ

Note that the form of this is a matrix sandwiched between a matrix and its inverse. The complicated projection T is said to be **similar** to the simple projection P_{x_1}.

Note that this only makes sense if we’re mapping from R^n back to itself.

Similarity

A linear transformation $T : R^n \to R^n$ is said to be **similar** to a linear transformation $S : R^n \to R^n$ if there exists an invertible linear transformation $P : R^n \to R^n$ such that

$$T = P^{-1} \circ S \circ P.$$

Likewise, an $n \times n$ matrix A is said to be **similar** to an $n \times n$ matrix B, if there exists an invertible $n \times n$ matrix C such that

$$A = C^{-1}BC.$$

Note that this can be viewed either direction since $T = P^{-1} \circ S \circ P$ and $A = C^{-1}BC$ imply

$$S = P \circ T \circ P^{-1} \quad \text{and} \quad B = CAC^{-1}$$

Using Similarity

Consider the matrix $A = \begin{bmatrix} -8 & -3 \\ 18 & 7 \end{bmatrix}$. Suppose we want to compute A^9.

$$A^9 = AAAAAAAAA = \underbrace{\begin{bmatrix} -8 & -3 \\ 18 & 7 \end{bmatrix}\begin{bmatrix} -8 & -3 \\ 18 & 7 \end{bmatrix}\cdots\begin{bmatrix} -8 & -3 \\ 18 & 7 \end{bmatrix}}_{\text{nine factors of } A.}$$

Compare that to computing D^9 if $D = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$.

$$D^2 = DD = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} (-2)^2 & 0 \\ 0 & 1^2 \end{bmatrix}$$

$$D^3 = D^2 D = \begin{bmatrix} (-2)^2 & 0 \\ 0 & 1^2 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} (-2)^3 & 0 \\ 0 & 1^3 \end{bmatrix}$$

$$D^n = \begin{bmatrix} (-2)^n & 0 \\ 0 & 1^n \end{bmatrix}$$

$$A = \begin{bmatrix} -8 & -3 \\ 18 & 7 \end{bmatrix}, \quad D = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \quad C^{-1} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

What if we know that $D = C^{-1}AC$ which means that $A = CDC^{-1}$?

Show that $D^2 = C^{-1}A^2C$ and $D^3 = C^{-1}A^3C$.

$$D^2 = DD = (C^{-1}AC)(C^{-1}AC)$$
$$= C^{-1}A(CC^{-1})AC$$
$$= C^{-1}AI_2AC = C^{-1}AAC = C^{-1}A^2C$$

$$D^3 = D^2D = (C^{-1}A^2C)(C^{-1}AC)$$
$$= C^{-1}A^2(CC^{-1})AC$$
$$= C^{-1}A^2I_2AC$$

$$= C^{-1}A^2 A C = C^{-1}A^3 C$$

If $D = C^{-1}AC$, then $D^n = C^{-1}A^n C$

Powers of Similar Matrices

If A and B are similar matrices, with $B = C^{-1}AC$ for some invertible matrix C, then for every integer $n \geq 1$

$$B^n = C^{-1}A^nC.$$

This means that $A^9 = CD^9C^{-1}$. That's two matrix multiplications instead of eight matrix multiplications.

$$A = \begin{bmatrix} -8 & -3 \\ 18 & 7 \end{bmatrix}, \quad D = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \quad C^{-1} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

$$D = C^{-1} A C \quad \Rightarrow \quad CD = C C^{-1} A C$$

$$CD = AC$$

$$CDC^{-1} = ACC^{-1}$$

$$CDC^{-1} = A.$$

$$A^9 = C D^9 C^{-1}$$

$$= \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} (-2)^9 & 0 \\ 0 & 1^9 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \cdot \begin{bmatrix} -512 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 3(-512) & -512 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3(-512)-2 & -512-1 \\ 6(512)+6 & 2(512)+3 \end{bmatrix}$$

$$= \begin{bmatrix} -1538 & -513 \\ 3078 & 1027 \end{bmatrix}$$