November 18 Math 2306 sec. 51 Fall 2022

Section 17: Fourier Series: Trigonometric Series

Consider the following problem:

An undamped spring mass system has a mass of 2 kg attached to a spring with spring constant 128 N/m. The mass is driven by an external force $f(t) = 2t$ for $-1 < t < 1$ that is 2-periodic so that $f(t+2) = f(t)$ for all $t > 0$.

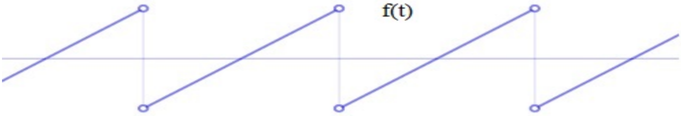

Figure: $$2\frac{d^2x}{dt^2} + 128x = f(t)$$

Common Models of Periodic Sources (e.g. Voltage)

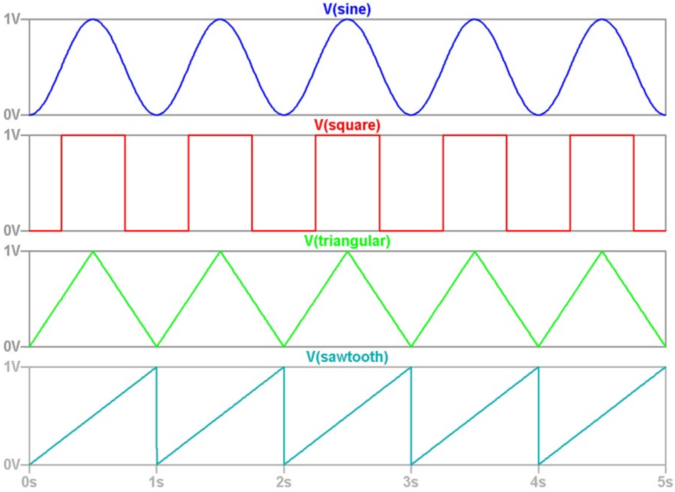

Figure: We'd like to solve, or at least approximate solutions, to ODEs and PDEs with periodic *right hand sides*.

Series Representations for Functions

The goal is to represent a function by a series

$$f(x) = \sum_{n=1}^{\infty} (\text{some simple functions})$$

In calculus, you saw power series $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ where the simple functions were powers $(x-c)^n$.

Here, you will see how some functions can be written as series of trigonometric functions

$$f(x) = \sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx)$$

We'll move the $n = 0$ to the front before the rest of the sum.

Some Preliminary Concepts

Suppose two functions f and g are integrable on the interval $[a, b]$. We define the **inner product** of f and g on $[a, b]$ as

$$\langle f, g \rangle = \int_a^b f(x)g(x)\,dx.$$

We say that f and g are **orthogonal** on $[a, b]$ if

$$\langle f, g \rangle = 0.$$

The product depends on the interval, so the orthogonality of two functions depends on the interval.

Properties of an Inner Product

Let f, g, and h be integrable functions on the appropriate interval and let c be any real number. The following hold

(i) $\langle f, g\rangle = \langle g, f\rangle$

(ii) $\langle f, g+h\rangle = \langle f, g\rangle + \langle f, h\rangle$

(iii) $\langle cf, g\rangle = c\langle f, g\rangle$

(iv) $\langle f, f\rangle \geq 0$ and $\langle f, f\rangle = 0$ if and only if $f = 0$

Orthogonal Set

A set of functions $\{\phi_0(x), \phi_1(x), \phi_2(x), \ldots\}$ is said to be **orthogonal** on an interval $[a, b]$ if

$$\langle \phi_m, \phi_n \rangle = \int_a^b \phi_m(x)\phi_n(x)\,dx = 0 \quad \text{whenever} \quad m \neq n.$$

Note that any function $\phi(x)$ that is not identically zero will satisfy

$$\langle \phi, \phi \rangle = \int_a^b \phi^2(x)\,dx > 0.$$

Hence we define the **square norm** of ϕ (on $[a, b]$) to be

$$\|\phi\| = \sqrt{\int_a^b \phi^2(x)\,dx}.$$

An Orthogonal Set of Functions

Consider the set of functions

$$\{1, \cos x, \cos 2x, \cos 3x, \dots, \sin x, \sin 2x, \sin 3x, \dots\} \quad \text{on} \quad [-\pi, \pi].$$

Evaluate $\langle \cos(nx), 1 \rangle$ and $\langle \sin(mx), 1 \rangle$.

$a = -\pi$
$b = \pi$

$$\langle \cos(nx), 1 \rangle = \int_{-\pi}^{\pi} \cos(nx) \cdot 1 \, dx$$

$$= \frac{1}{n} \sin(nx) \Big|_{-\pi}^{\pi}$$

$$= \frac{1}{n} \sin(n\pi) - \frac{1}{n} \sin(-n\pi)$$

$$= \frac{1}{n}(0) - \frac{1}{n}(0) = 0$$

$*\ \sin(n\pi) = 0$ for any integer n

$\langle \cos(n\pi), 1\rangle = 0$ for all n

all $\cos(nx)$ are orthogonal to 1 on $[-\pi, \pi]$.

$$\langle \sin(mx), 1\rangle = \int_{-\pi}^{\pi} \sin(mx)\,1\,dx$$

$$= \frac{-1}{m}\cos(mx)\Big|_{-\pi}^{\pi}$$

$$= \frac{-1}{m}\cos(m\pi) - \frac{-1}{m}\cos(-m\pi)$$

$\cos(-\theta) = \cos\theta$

$$= \frac{-1}{m}\cos(m\pi) + \frac{1}{m}\cos(m\pi) = 0$$

$\langle \sin(mx), 1\rangle = 0$ for all m so $\sin(mx)$ is orthogonal to 1 on $[-\pi, \pi]$.

An Orthogonal Set of Functions

Consider the set of functions

$$\{1, \cos x, \cos 2x, \cos 3x, \ldots, \sin x, \sin 2x, \sin 3x, \ldots\} \quad \text{on} \quad [-\pi, \pi].$$

It can easily be verified that

$$\int_{-\pi}^{\pi} \cos nx\, dx = 0 \quad \text{and} \quad \int_{-\pi}^{\pi} \sin mx\, dx = 0 \quad \text{for all} \quad n, m \geq 1,$$

$$\int_{-\pi}^{\pi} \cos nx\, \sin mx\, dx = 0 \quad \text{for all} \quad m, n \geq 1, \quad \text{and}$$

$$\int_{-\pi}^{\pi} \cos nx\, \cos mx\, dx = \int_{-\pi}^{\pi} \sin nx\, \sin mx\, dx = \begin{cases} 0, & m \neq n \\ \pi, & n = m \end{cases},$$

An Orthogonal Set of Functions on $[-\pi, \pi]$

These integral values indicated that the set of functions

$$\{1, \cos x, \cos 2x, \cos 3x, \ldots, \sin x, \sin 2x, \sin 3x, \ldots\}$$

is an orthogonal set on the interval $[-\pi, \pi]$.

Key Point: This means that if we take any two functions f and g **from this set**, then

$$\int_{-\pi}^{\pi} f(x)g(x)\,dx = 0 \quad \text{if } f \text{ and } g \text{ are different functions!}$$

Fourier Series

Suppose $f(x)$ is defined for $-\pi < x < \pi$. We would like to know how to write f as a series **in terms of sines and cosines.**

Task: Find coefficients (numbers) $a_0, a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ such that[1]

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right).$$

[1] We'll write $\frac{a_0}{2}$ as opposed to a_0 purely for convenience.

Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)\,.$$

The question of convergence naturally arises when we wish to work with infinite series. To highlight convergence considerations, some authors prefer not to use the equal sign when expressing a Fourier series and instead write

$$f(x) \sim \frac{a_0}{2} + \cdots$$

Herein, we'll use the equal sign with the understanding that equality may not hold at each point.

Convergence will be address later.

Finding an Example Coefficient

Let's find the coefficient b_4.

Start with the series $f(x) = \dfrac{a_0}{2} + \sum\limits_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$, and multiply both sides by $\sin(4x)$.

$$f(x)\sin(4x) = \frac{a_0}{2}\sin(4x) + \sum_{n=1}^{\infty} \left(a_n \cos nx \sin(4x) + b_n \sin nx \sin(4x)\right).$$

Now, integrate from $-\pi$ to π

$$\int_{-\pi}^{\pi} f(x)\sin(4x)\,dx = \int_{-\pi}^{\pi} \frac{a_0}{2}\sin(4x)\,dx + \int_{-\pi}^{\pi} \left(\sum_{n=1}^{\infty} a_n \cos(nx)\sin(4x) + b_n \sin(nx)\sin(4x)\right) dx$$

$$\int_{-\pi}^{\pi} f(x)\sin(4x)\,dx =$$

$$\underbrace{\frac{a_0}{2}\int_{-\pi}^{\pi}\sin(4x)\,dx}_{=0} + \sum_{n=1}^{\infty}\left(a_n \underbrace{\int_{-\pi}^{\pi}\cos(nx)\sin(4x)\,dx}_{=0} + b_n\int_{-\pi}^{\pi}\sin(nx)\sin(4x)\,dx\right)$$

$$\langle \sin 4x, 1\rangle = \int_{-\pi}^{\pi}\sin(4x)\,dx = 0$$

$$\langle \cos(nx), \sin(4x)\rangle = 0$$

$$\int_{-\pi}^{\pi} f(x)\sin(4x)\,dx = \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi}\sin(nx)\sin(4x)\,dx$$

$$\int_{-\pi}^{\pi} \sin(nx)\sin(4x)\,dx = \begin{cases} 0, & n \neq 4 \\ \pi, & n = 4 \end{cases}$$

$$\int_{-\pi}^{\pi} f(x)\sin(4x) = \pi b_4$$

$$\boxed{\text{So} \quad b_4 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(4x)\,dx}$$

Finding Fourier Coefficients

Note that there was nothing special about seeking the 4^{th} sine coefficient b_4. We could have just as easily sought b_m for any positive integer m. We would simply start by introducing the factor $\sin(mx)$.

Moreover, using the same orthogonality property, we could pick on the a's by starting with the factor $\cos(mx)$—including the constant term since $\cos(0 \cdot x) = 1$. The only minor difference we want to be aware of is that

$$\int_{-\pi}^{\pi} \cos^2(mx)\,dx = \begin{cases} 2\pi, & m = 0 \\ \pi, & m \geq 1 \end{cases}$$

Careful consideration of this sheds light on why it is conventional to take the constant to be $\frac{a_0}{2}$ as opposed to just a_0.

The Fourier Series of $f(x)$ on $(-\pi, \pi)$

The **Fourier series** of the function f defined on $(-\pi, \pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right).$$

Where

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx, \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad \text{and} \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \end{aligned}$$