# November 18 Math 2306 sec. 52 Fall 2022

## Section 17: Fourier Series: Trigonometric Series

### Consider the following problem:

An undamped spring mass system has a mass of 2 kg attached to a spring with spring constant 128 N/m. The mass is driven by an external force $f(t) = 2t$ for $-1 < t < 1$ that is 2-periodic so that $f(t+2) = f(t)$ for all $t > 0$.

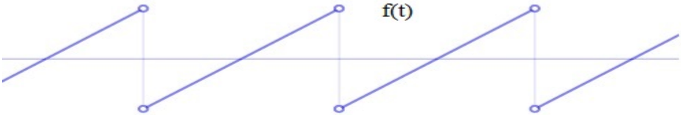


Figure: $$2\frac{d^2x}{dt^2} + 128x = f(t)$$

# Common Models of Periodic Sources (e.g. Voltage)

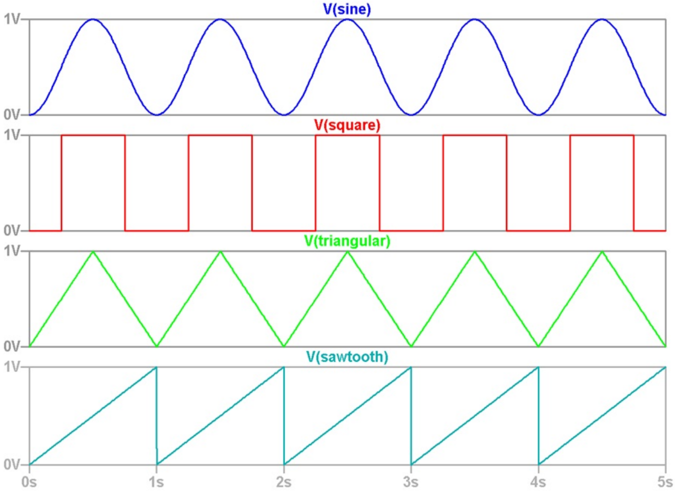


Figure: We'd like to solve, or at least approximate solutions, to ODEs and PDEs with periodic *right hand sides*.

# Series Representations for Functions

The goal is to represent a function by a series

$$f(x) = \sum_{n=1}^{\infty} (\text{some simple functions})$$

In calculus, you saw power series $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ where the simple functions were powers $(x-c)^n$.

Here, you will see how some functions can be written as series of trigonometric functions

$$f(x) = \sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx)$$

We'll move the $n = 0$ to the front before the rest of the sum.

# Some Preliminary Concepts

Suppose two functions $f$ and $g$ are integrable on the interval $[a, b]$. We define the **inner product** of $f$ and $g$ on $[a, b]$ as

$$\langle f, g \rangle = \int_a^b f(x)g(x)\,dx.$$

We say that $f$ and $g$ are **orthogonal** on $[a, b]$ if

$$\langle f, g \rangle = 0.$$

The product depends on the interval, so the orthogonality of two functions depends on the interval.

# Properties of an Inner Product

Let $f$, $g$, and $h$ be integrable functions on the appropriate interval and let $c$ be any real number. The following hold

(i) $\langle f, g\rangle = \langle g, f\rangle$

(ii) $\langle f, g+h\rangle = \langle f, g\rangle + \langle f, h\rangle$

(iii) $\langle cf, g\rangle = c\langle f, g\rangle$

(iv) $\langle f, f\rangle \geq 0$ and $\langle f, f\rangle = 0$ if and only if $f = 0$

# Orthogonal Set

A set of functions $\{\phi_0(x), \phi_1(x), \phi_2(x), \ldots\}$ is said to be **orthogonal** on an interval $[a, b]$ if

$$\langle \phi_m, \phi_n \rangle = \int_a^b \phi_m(x)\phi_n(x)\, dx = 0 \quad \text{whenever} \quad m \neq n.$$

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Note that any function $\phi(x)$ that is not identically zero will satisfy

$$\langle \phi, \phi \rangle = \int_a^b \phi^2(x)\, dx > 0.$$

Hence we define the **square norm** of $\phi$ (on $[a, b]$) to be

$$\|\phi\| = \sqrt{\int_a^b \phi^2(x)\, dx}.$$

# An Orthogonal Set of Functions

Consider the set of functions

$$\{1, \cos x, \cos 2x, \cos 3x, \ldots, \sin x, \sin 2x, \sin 3x, \ldots\} \quad \text{on} \quad [-\pi, \pi].$$

Evaluate $\langle \cos(nx), 1\rangle$ and $\langle \sin(mx), 1\rangle$.

$a = -\pi$
$b = \pi$

$$\langle \cos(nx), 1\rangle = \int_{-\pi}^{\pi} \cos(nx) \cdot 1 \, dx$$

$$= \frac{1}{n} \sin(nx) \Big|_{-\pi}^{\pi} = \frac{1}{n} \sin(n\pi) - \frac{1}{n} \sin(-n\pi)$$

$$= \frac{1}{n}(0) - \frac{1}{n}(0) = 0$$

$\cos(nx)$ is orthogonal to $1$ on $[-\pi, \pi]$ for all $n$.

$$\langle \sin(mx), 1 \rangle = \int_{-\pi}^{\pi} \sin(mx)\cdot 1 \, dx$$

$$= \frac{-1}{m} \cos(mx) \Big|_{-\pi}^{\pi}$$

$$= \frac{-1}{m} \cos(m\pi) - \frac{-1}{m} \cos(-m\pi)$$

$\cos(-\theta) = \cos\theta$

$$= \frac{-1}{m} \cos(m\pi) + \frac{1}{m} \cos(m\pi) = 0$$

Hence $\sin(mx)$ is orthogonal to $1$ on $[-\pi, \pi]$ for all $m$.

# An Orthogonal Set of Functions

Consider the set of functions

$$\{1, \cos x, \cos 2x, \cos 3x, \ldots, \sin x, \sin 2x, \sin 3x, \ldots\} \quad \text{on} \quad [-\pi, \pi].$$

It can easily be verified that

$$\int_{-\pi}^{\pi} \cos nx\, dx = 0 \quad \text{and} \quad \int_{-\pi}^{\pi} \sin mx\, dx = 0 \quad \text{for all} \quad n, m \geq 1,$$

$$\int_{-\pi}^{\pi} \cos nx\, \sin mx\, dx = 0 \quad \text{for all} \quad m, n \geq 1, \quad \text{and}$$

$$\int_{-\pi}^{\pi} \cos nx\, \cos mx\, dx = \int_{-\pi}^{\pi} \sin nx\, \sin mx\, dx = \begin{cases} 0, & m \neq n \\ \pi, & n = m \end{cases},$$

# An Orthogonal Set of Functions on $[-\pi, \pi]$

These integral values indicated that the set of functions

$$\{1, \cos x, \cos 2x, \cos 3x, \ldots, \sin x, \sin 2x, \sin 3x, \ldots\}$$

is an orthogonal set on the interval $[-\pi, \pi]$.

**Key Point:** This means that if we take any two functions $f$ and $g$ **from this set**, then

$$\int_{-\pi}^{\pi} f(x)g(x)\,dx = 0 \quad \text{if } f \text{ and } g \text{ are different functions!}$$

# Fourier Series

Suppose $f(x)$ is defined for $-\pi < x < \pi$. We would like to know how to write $f$ as a series **in terms of sines and cosines.**

**Task:** Find coefficients (numbers) $a_0$, $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ such that[1]

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos nx + b_n \sin nx \right).$$

[1] We'll write $\frac{a_0}{2}$ as opposed to $a_0$ purely for convenience.

# Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos nx + b_n \sin nx \right).$$

The question of convergence naturally arises when we wish to work with infinite series. To highlight convergence considerations, some authors prefer not to use the equal sign when expressing a Fourier series and instead write

$$f(x) \sim \frac{a_0}{2} + \cdots$$

Herein, we'll use the equal sign with the understanding that equality may not hold at each point.

Convergence will be address later.

## Finding an Example Coefficient

Let's find the coefficient $b_4$.

Start with the series $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$, and multiply both sides by $\sin(4x)$.

$$f(x)\sin(4x) = \frac{a_0}{2}\sin(4x) + \sum_{n=1}^{\infty}(a_n \cos nx \sin(4x) + b_n \sin nx \sin(4x)).$$

Integrate from $-\pi$ to $\pi$

$$\int_{-\pi}^{\pi} f(x)\sin(4x)\,dx =$$

$$\int_{-\pi}^{\pi} \frac{a_0}{2}\sin(4x)\,dx + \int_{-\pi}^{\pi}\left(\sum_{n=1}^{\infty} a_n \cos(nx)\sin(4x) + b_n \sin(nx)\sin(4x)\right)dx$$

$$\int_{-\pi}^{\pi} f(x)\sin(4x)\,dx =$$

$$\underbrace{\frac{a_0}{2}\int_{-\pi}^{\pi}\sin(4x)\,dx}_{0} + \sum_{n=1}^{\infty}\left(a_n \underbrace{\int_{-\pi}^{\pi}\cos(nx)\sin(4x)\,dx}_{0} + b_n\int_{-\pi}^{\pi}\sin(nx)\sin(4x)\,dx\right)$$

$$\langle \sin(4x), 1\rangle = 0 \qquad \langle \cos(nx), \sin(4x)\rangle = 0$$

$$\int_{-\pi}^{\pi} f(x)\sin(4x)\,dx = \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi}\sin(nx)\sin(4x)\,dx$$

$$\langle \sin(nx), \sin(4x)\rangle = \begin{cases} 0, & n \neq 4 \\ \pi, & n = 4 \end{cases}$$

$$\int_{-\pi}^{\pi} f(x) \sin(4x)\,dx = b_4 \pi$$

$$\Rightarrow \quad b_4 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(4x)\,dx$$

# Finding Fourier Coefficients

Note that there was nothing special about seeking the $4^{th}$ sine coefficient $b_4$. We could have just as easily sought $b_m$ for any positive integer $m$. We would simply start by introducing the factor $\sin(mx)$.

Moreover, using the same orthogonality property, we could pick on the $a$'s by starting with the factor $\cos(mx)$—including the constant term since $\cos(0 \cdot x) = 1$. The only minor difference we want to be aware of is that

$$\int_{-\pi}^{\pi} \cos^2(mx)\,dx = \begin{cases} 2\pi, & m = 0 \\ \pi, & m \geq 1 \end{cases}$$

Careful consideration of this sheds light on why it is conventional to take the constant to be $\frac{a_0}{2}$ as opposed to just $a_0$.

# The Fourier Series of $f(x)$ on $(-\pi, \pi)$

The **Fourier series** of the function $f$ defined on $(-\pi, \pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right).$$

Where

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx, \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad \text{and} \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx
\end{aligned}$$