November 18 Math 2306 sec. 53 Fall 2024

Section 16: Laplace Transforms of Derivatives and IVPs

Use Laplace Transforms to Solve and IVP

- Start with constant coefficient IVP with IC at $t = 0$. For example,

$$ay'' + by' + cy = g(t), \quad y(0) = y_0, \quad y'(0) = y_1.$$

- Let $Y(s) = \mathscr{L}\{y(t)\}$ and take the transform of both sides of the ODE using any necessary results.
- Sub in the initial conditions where they appear in the transformed equation.
- Use basic algebra to isolate the transform $Y(s)$.
- Using whatever algebra or function identities that are needed, take the inverse transform to obtain the solution

$$y(t) = \mathscr{L}^{-1}\{Y(s)\}.$$

An IVP with Piecewise Input

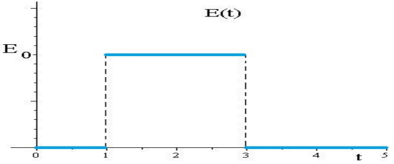

Figure: $L\dfrac{di}{dt} + Ri = E(t)$, $i(0) = 0$ where $L = 1$, $R = 10$ and $E(t)$ is shown.

Last time, we solved this IVP using the Laplace transform and got the current

$$i(t) = \frac{E_0}{10}\left(1 - e^{-10(t-1)}\right)\mathscr{U}(t-1) - \frac{E_0}{10}\left(1 - e^{-10(t-3)}\right)\mathscr{U}(t-3)$$

$$= \begin{cases} 0, & 0 \le t < 1 \\ \dfrac{E_0}{10}\left(1 - e^{-10(t-1)}\right), & 1 \le t < 3 \\ \dfrac{E_0}{10}\left(e^{-10(t-3)} - e^{-10(t-1)}\right), & t \ge 3 \end{cases}$$

The Unit Impulse

Now, suppose we wish to consider this circuit, but are interested in taking the limit as the interval over which the voltage is applied.

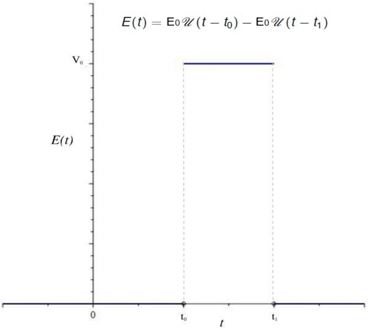

Figure: The charge on the capacitor satisfies the differential equation $L\dfrac{di}{dt} + Ri = E_0\mathscr{U}(t-t_0) - E_0\mathscr{U}(t-t_1)$

The Unit Impulse

In engineering applications, it is useful to have a model of a force (or signal) that is applied over an infinitesimal time interval. That is, we would like to model this process in the limit $t_1 \to t_0$ while keeping the total *impulse*[1] of the force fixed.

We can build such a model by considering rectangular[2] functions and reducing the width while keeping the area fixed.

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[1] Impulse is a measure of the effect of a force over a time interval (e.g. force times time). It is the change in momentum.

[2] The shape doesn't have to be a rectangle. It could be triangles, or a hump of a cosine, or something else.

The Unit Impulse

In order to build up to the definition of our unit impulse, we introduce the family of piecewise constant, rectangular functions $R_\epsilon(t) = \begin{cases} \frac{1}{2\epsilon}, & |t| < \epsilon \\ 0, & |t| > \epsilon \end{cases}$.

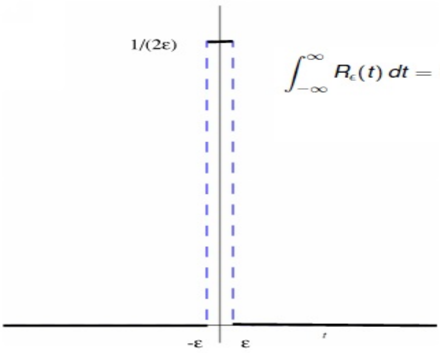
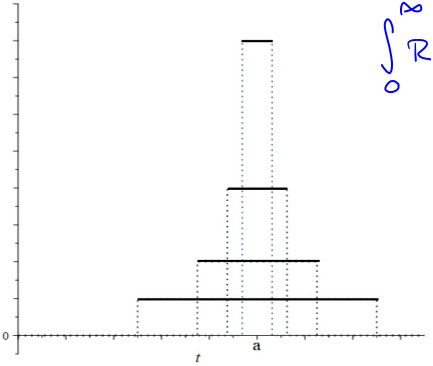


Figure: The value of ϵ determines the height and width of the rectangle. But for every $\epsilon > 0$, the integral of R_ϵ over the real line is 1.

Unit Impulse

We can place the peak of our rectangular functions at $t = a$ by considering $R_\epsilon(t-a)$. We see that at the value ϵ gets smaller, the height of the rectangle increases while the width decreases in such a way that the area remains constant at 1.



$$\int_0^\infty R_\varepsilon(t-a)\,dt = 1$$

Figure: Plots of $R_\epsilon(t-a) = \begin{cases} \frac{1}{2\epsilon}, & |t-a| < \epsilon \\ 0, & |t-a| > \epsilon \end{cases}$ for a few different values of ϵ.

Unit Impulse

The Dirac delta *function*, denoted by $\delta(\cdot)$, models an instantaneous force at time $t = a$ with unit impulse. One way to conceptualize this *function* is as the limit

$$\delta(t-a) = \lim_{\epsilon \to 0} R_\epsilon(t-a).$$

This idea will suffice for our practical interest in the Dirac delta.

The actual limit is stated in terms of integrals. That is, we define δ by

$$\int_{-\infty}^{\infty} \delta(t-a)g(t)\,dt = \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} R_\epsilon(t-a)g(t)\,dt$$

where equality holds for all *nice* functions g. (See section 15 of the complete lecture text for a more thorough treatment.)

Unit Impulse $\delta(t-a)$

The Dirac delta function is a limit of traditional functions, but it isn't really a function (in the input-output sense). It is an example of what is called a *generalized function*, a *functional*, or a *distribution*. It is a mathematical object whose properties are defined in combination with integration. We can think of it as acting on continuous functions in specific ways.

If $a>0$

$$\int_0^\infty \delta(t-a)\,dt = 1$$

The following hold:

- $\displaystyle\int_{-\infty}^{\infty} \delta(t-a)\,dt = 1$ for any real number a.
- $\displaystyle\int_{-\infty}^{\infty} \delta(t-a)f(t)\,dt = f(a)$ if f is continuous at a.
- $\mathscr{L}\{\delta(t-a)\} = e^{-as}$ for any constant $a \geq 0$.
- In the sense of distributions, it is related to $\mathscr{U}$ via $\dfrac{d}{dt}\mathscr{U}(t-a) = \delta(t-a)$.

$$\int_0^\infty \delta(t-a)f(t)\,dt = f(a)$$

Delta as a Model of a Unit Impulse

A function $f(t) = f_0\delta(t-a)$ can be used to model a force of impulse f_0 applied instantaneously at the time $t = a$.

For example, suppose our LR series circuit has zero applied voltage for $t \neq t_0$. A switch is closed and opened immediately to apply a voltage E_0 at $t = t_0$. The differential equation modeling the charge on the capacitor is given by

$$L\frac{di}{dt} + Ri = E_0\delta(t-t_0).$$

Remark

We can't work with the Dirac delta function the way we might work with other forcing functions (e.g., exponentials or sines and cosines). But we do know what the Laplace transform of $\delta(t-t_0)$ is, so we will be able to solve IVPs that involve differential equations of the form shown here.

Solve the IVP using the Laplace Transform

A 1 kg mass is suspended from a spring with spring constant 10 N/m. A damper induces damping of 6 N per m/sec of velocity. The object starts from rest at equilibrium. At time $t = 1$ second, a unit impulse force is applied to the object. Determine the object's position for $t > 0$.

The corresponding IVP for the situation described is

$$x'' + 6x' + 10x = \delta(t-1), \quad x(0) = 0, \quad x'(0) = 0$$

Let $X(s) = \mathcal{L}\{x(t)\}$.

$$\mathcal{L}\{x'' + 6x' + 10x\} = \mathcal{L}\{\delta(t-1)\}.$$

$$\mathcal{L}\{x''\} + 6\,\mathcal{L}\{x'\} + 10\,\mathcal{L}\{x\} = \mathcal{L}\{\delta(t-1)\}$$

Remember the model is $mx'' + bx' + kx = f(t)$ with initial position $x(0)$ and initial velocity $x'(0)$.

$$s^2 X(s) - s\,\underset{0}{\underbrace{x(0)}} - \underset{0}{\underbrace{x'(0)}} + 6\left(sX(s) - \underset{0}{\underbrace{x(0)}}\right) + 10X(s) = e^{-1s}$$

$$s^2 X(s) + 6sX(s) + 10X(s) = e^{-s}$$

$$(s^2 + 6s + 10)X(s) = e^{-s}$$

Char. poly

$$x'' + 6x' + 10x = \delta(t-1),$$

$$X(s) = \frac{e^{-s}}{s^2 + 6s + 10}$$

$$X(s) = e^{-s} \frac{1}{s^2 + 6s + 10}$$

we need $x(t) = \mathcal{L}^{-1}\{X(s)\}$. For this, we need $f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+6s+10}\right\}$.

Since $s^2+6s+10$ doesn't factor, we complete the square.

$$s^2+6s+9-9+10 = (s+3)^2+1$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2+1}\right\} = e^{-3t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$\frac{1}{s^2+1}$ with $s-(-3)$ in place of s

$$= e^{-3t}\sin t$$

$$X(s) = e^{-s} \frac{1}{s^2+6s+10}$$

Our solution will be $f(t-1)u(t-1)$

The position $x(t) = \mathcal{L}^{-1}\{X(s)\}$ is

$$\boxed{x(t) = e^{-3(t-1)} \sin(t-1)\, u(t-1).}$$

$$x'' + 6x' + 10x = \delta(t-1), \quad x(0) = 0, \quad x'(0) = 0$$

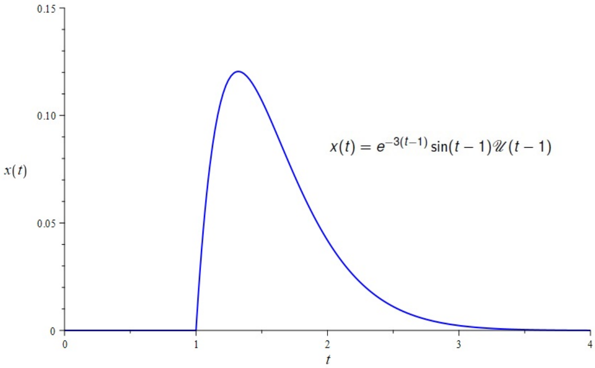


Figure: Graph of the solution to the IVP with unit impulse external force at $t = 1$.

Transfer Function & Impulse Response

$$ay'' + by' + cy = g(t), \tag{1}$$

Definition

The function $H(s) = \dfrac{1}{as^2 + bs + c}$ is called the **transfer function** for the differential equation (2).

Remark 1: The transfer function is the Laplace transform of the solution to the IVP

$$ay'' + by' + cy = \delta(t), \quad y(0) = 0, \quad y'(0) = 0.$$

Transfer Function & Impulse Response

$$ay'' + by' + cy = g(t), \tag{2}$$

Definition

The **impulse response** function, $h(t)$, for the differential equation (2) is the inverse Laplace transform of the transfer function

$$h(t) = \mathscr{L}^{-1}\{H(s)\} = \mathscr{L}^{-1}\left\{\frac{1}{as^2 + bs + c}\right\}.$$

Remark 2: The impulse response is the solution to the IVP

$$ay'' + by' + cy = \delta(t), \quad y(0) = 0, \quad y'(0) = 0.$$

Convolution

$$ay'' + by' + cy = g(t), \quad y(0) = y_0, \quad y'(0) = y_1$$

Recall the **zero state response** is $\mathscr{L}^{-1}\left\{\dfrac{G(s)}{as^2 + bs + c}\right\}$. We can write this as

$$\mathscr{L}^{-1}\left\{G(s)H(s)\right\},$$

where H is the transfer function.

The Zero State Response is the convolution of g and the impulse response h.

If the impulse response is $h(t)$, then the zero state response can be written in terms of a convolution as

$$\mathscr{L}^{-1}\left\{G(s)H(s)\right\} = \int_0^t g(\tau)h(t-\tau)\,d\tau$$

Solving a System

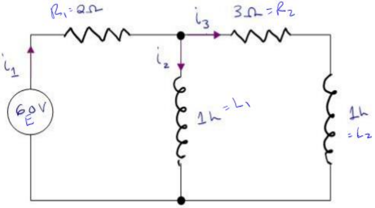
We can solve a system of ODEs using Laplace transforms. Here, we'll consider systems that are

- linear,
- having initial conditions at $t = 0$, and
- constant coefficient.

Let's see it in action (i.e. with a couple of examples).

Example

We'll solve this next time.



$$
\begin{aligned}
L_1 i_2' + R_1(i_2+i_3) &= E \\
L_2 i_3' + R_1(i_2+i_3) + R_2 i_3 &= E
\end{aligned}
$$

Figure: If we label current i_2 as $x(t)$ and current i_3 as $y(t)$, we get the system of equations below. (Assuming $i_1(0) = 0$.)

Solve the system of equations

$$
\begin{aligned}
\frac{dx}{dt} &= -2x - 2y + 60, \quad x(0) = 0 \\
\frac{dy}{dt} &= -2x - 5y + 60, \quad y(0) = 0
\end{aligned}
$$