November 19 Math 2306 sec. 51 Fall 2021

Section 18: Sine and Cosine Series

We consider a function f defined on the interval $(0, p)$ and extend it to be even or odd.

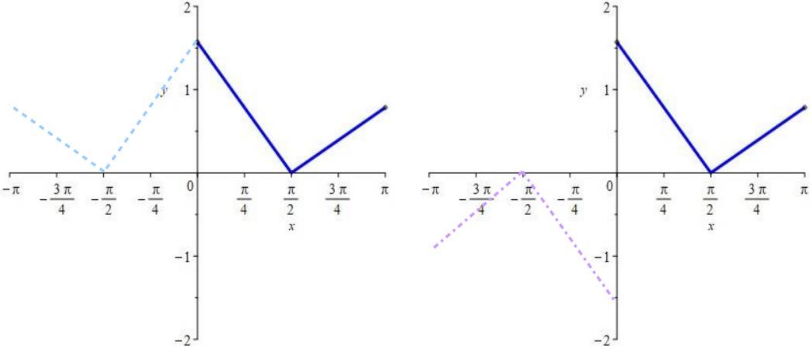


Figure: Some function f shown in dark blue with even and odd extensions.

Half Range Cosine Series

For f defined on $(0, p)$ we can *pretend* that f is defined on $(-p, p)$ by setting

$$f(-x) = f(x) \quad 0 < x < p.$$

Then we can define the

Half range cosine series $\quad f(x) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{p}\right)$

where $\quad a_0 = \dfrac{2}{p}\displaystyle\int_0^p f(x)\,dx \quad$ and $\quad a_n = \dfrac{2}{p}\displaystyle\int_0^p f(x)\cos\left(\frac{n\pi x}{p}\right)\,dx.$

Half Range Sine

Suppose f is defined on $(0, p)$. We can extend f to be defined on $(-p, p)$ by setting

$$f(-x) = -f(x) \quad 0 < x < p.$$

Then we can define the

$$\textbf{Half range sine series} \quad f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{p}\right)$$

$$\text{where} \quad b_n = \frac{2}{p}\int_0^p f(x)\sin\left(\frac{n\pi x}{p}\right)\,dx.$$

Find the Half Range Sine Series of f

$$f(x) = 2 - x, \quad 0 < x < 2$$

We worked this out and found the half-range cosine series

$$f(x) = \sum_{n=1}^{\infty} \frac{4}{n\pi} \sin\left(\frac{n\pi x}{2}\right).$$

Find the Half Range Cosine Series of f

$$f(x) = 2 - x, \quad 0 < x < 2$$ $p = 2$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right)$$

$$a_0 = \frac{2}{2}\int_0^2 f(x)\,dx = \int_0^2 (2-x)\,dx$$

$$= 2x - \frac{x^2}{2}\Big|_0^2 = 4 - 2 = 2$$

$$a_n = \frac{2}{2}\int_0^2 f(x)\cos\left(\frac{n\pi x}{2}\right)dx = \int_0^2 (2-x)\cos\left(\frac{n\pi x}{2}\right)dx$$

Int by parts

$$u = 2 - x, \qquad du = -dx$$

$$dv = \cos\left(\frac{n\pi x}{2}\right)dx \qquad v = \frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)$$

$$a_n = \frac{2}{n\pi}\underbrace{(2-x)\sin\left(\frac{n\pi x}{2}\right)\Big|_0^2}_{0} + \int_0^2 \frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{2}{n\pi}\left(\frac{-2}{n\pi}\right)\cos\left(\frac{n\pi x}{2}\right)\Big|_0^2$$

$$= \frac{-4}{n^2\pi^2}\left(\cos(n\pi) - \cos(0)\right)$$

$$= \frac{-4}{n^2\pi^2}\left((-1)^n - 1\right)$$

$$a_n = \frac{4}{n^2\pi^2}\left(1-(-1)^n\right), \quad a_0 = 2$$

Finally,

$$\boxed{f(x) = 1 + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\left(1-(-1)^n\right)\cos\left(\frac{n\pi x}{2}\right)}$$

Example Continued...

We have two different half range series:

$$\text{Half range sine:} \quad f(x) = \sum_{n=1}^{\infty} \frac{4}{n\pi} \sin\left(\frac{n\pi x}{2}\right)$$

$$\text{Half range cosine:} \quad f(x) = 1 + \sum_{n=1}^{\infty} \frac{4(1-(-1)^n)}{n^2\pi^2} \cos\left(\frac{n\pi x}{2}\right).$$

We have two different series representations for this function each of which converge to $f(x)$ on the interval $(0,2)$. The following plots show graphs of f along with partial sums of each of the series. When we plot over the interval $(-2,2)$ we see the two different symmetries. Plotting over a larger interval such as $(-6,6)$ we can see the periodic extensions of the two symmetries.

Plots of f with Half range series

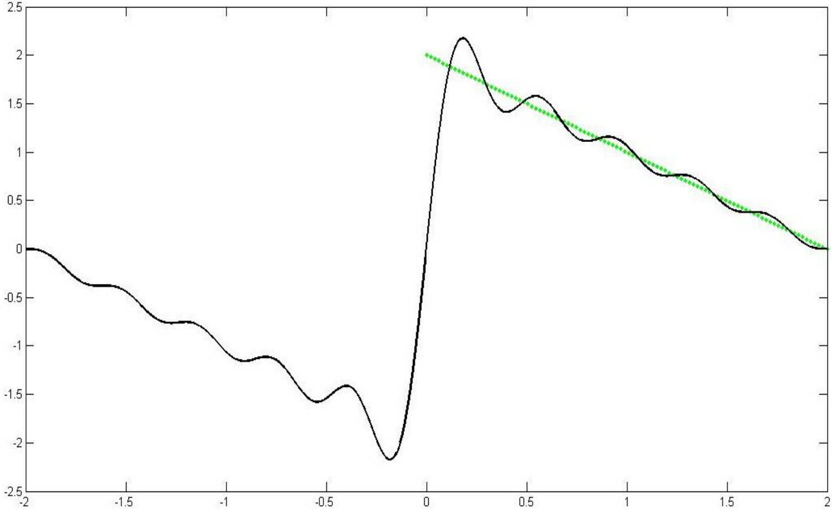


Figure: $f(x) = 2 - x$, $0 < x < 2$ with 10 terms of the sine series.

Plots of f with Half range series

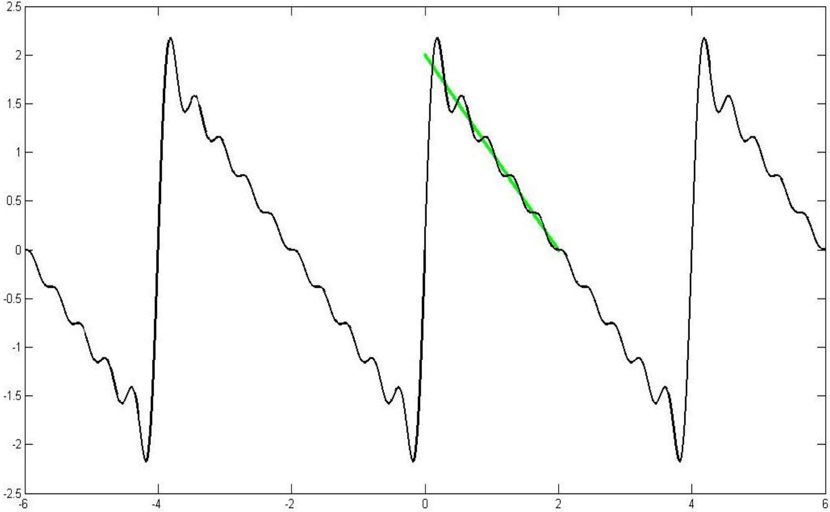


Figure: $f(x) = 2 - x$, $0 < x < 2$ with 10 terms of the sine series, and the series plotted over $(-6, 6)$

Plots of f with Half range series

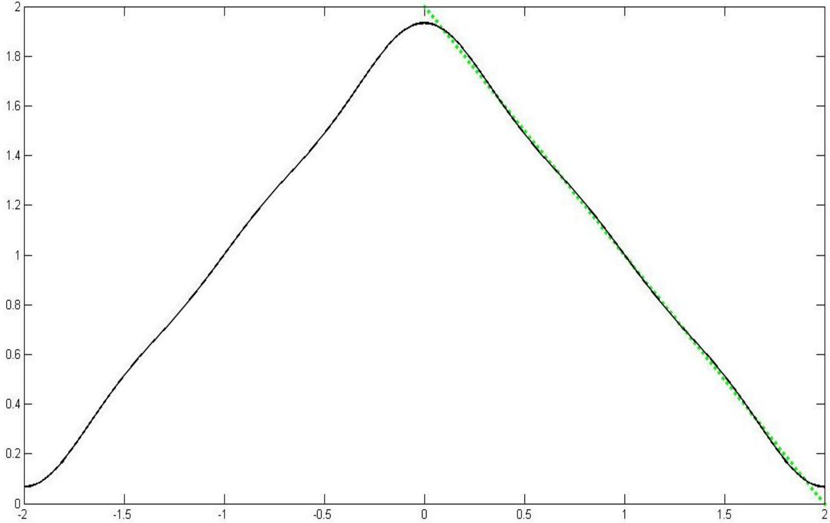


Figure: $f(x) = 2 - x$, $0 < x < 2$ with 5 terms of the cosine series.

Plots of f with Half range series

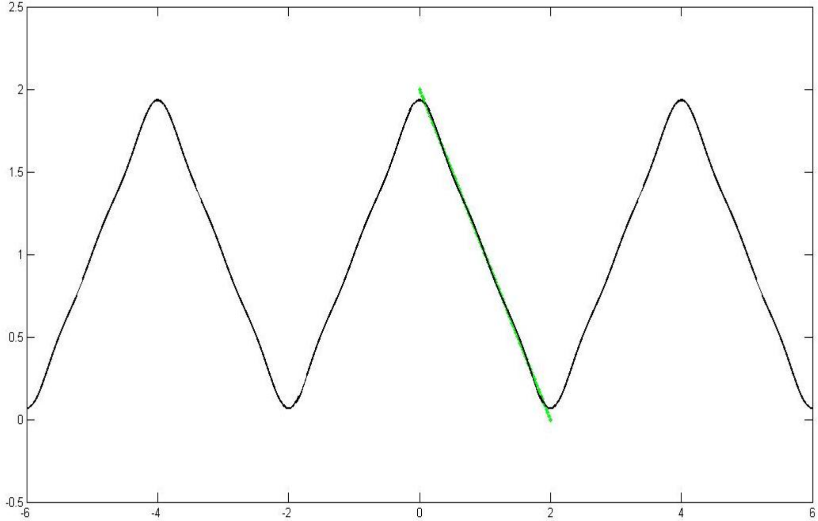
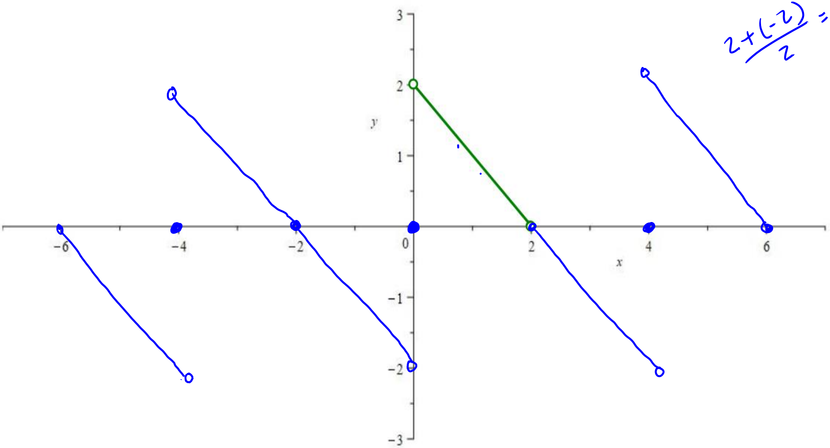


Figure: $f(x) = 2 - x$, $0 < x < 2$ with 5 terms of the cosine series, and the series plotted over $(-6, 6)$

Plot the Half-Range Sine Series

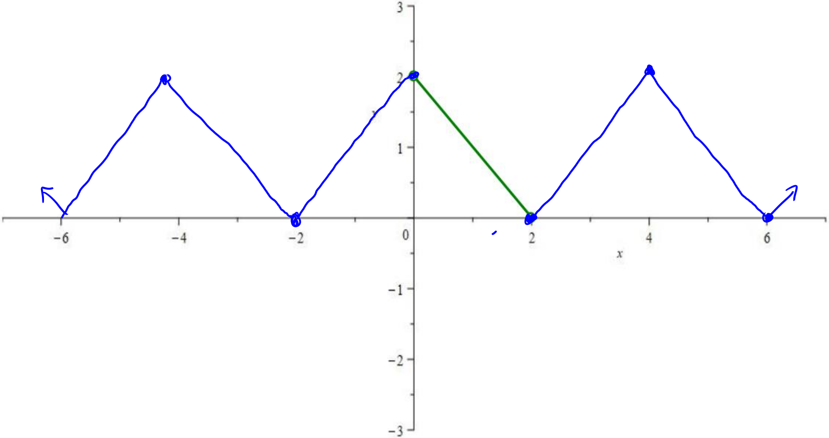
$$f(x) = 2 - x, \quad 0 < x < 2 \qquad f(x) = \sum_{n=1}^{\infty} \frac{4}{n\pi} \sin\left(\frac{n\pi x}{2}\right)$$

$$\frac{2+(-2)}{2} = 0$$



Plot the Half-Range Cosine Series

$$f(x) = 2 - x, \quad 0 < x < 2 \qquad f(x) = 1 + \sum_{n=1}^{\infty} \frac{4(1-(-1)^n)}{n^2\pi^2} \cos\left(\frac{n\pi x}{2}\right)$$



Solution of a Differential Equation

An undamped spring mass system has a mass of 2 kg attached to a spring with spring constant 128 N/m. The mass is driven by an external force $f(t) = 2t$ for $-1 < t < 1$ that is 2-periodic so that $f(t+2) = f(t)$ for all $t > 0$. Determine a particular solution x_p for the displacement for $t > 0$.

$mx'' + kx = f(t)$ $m=2,\quad k=128$

$2x'' + 128x = f(t) \Rightarrow x'' + 64x = \frac{1}{2}f(t)$

Let's expand f in a Fourier Series.

From Nov 15th, we have

$$f(t) = 2\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi t)$$

The ODE is

$$x'' + 64x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi t)$$

$$= \frac{2}{\pi}\sin(\pi t) - \frac{1}{\pi}\sin(2\pi t) + \frac{2}{3\pi}\sin(3\pi t) + \cdots$$

Using the method of undetermined coefficients we'll look for x_p in the form

$$x_p = \sum_{n=1}^{\infty} B_n \sin(n\pi t)$$

and assume that the series can be differentiated term by term.

Hence

$$X_p' = \sum_{n=1}^{\infty} \frac{d}{dt}\left(B_n \sin(n\pi t)\right)$$

$$= \sum_{n=1}^{\infty} B_n (n\pi) \cos(n\pi t)$$

$$X_p'' = \sum_{n=1}^{\infty} -B_n (n\pi)^2 \sin(n\pi t)$$

$$X_p = \sum_{n=1}^{\infty} B_n \sin(n\pi t)$$

$$x_p'' + 64x_p = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi t)$$

$$\sum_{n=1}^{\infty} -B_n (n\pi)^2 \sin(n\pi t) + 64 \sum_{n=1}^{\infty} B_n \sin(n\pi t)$$

$$= \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi t)$$

$$\sum_{n=1}^{\infty} \left(-(n\pi)^2 + 64\right) B_n \sin(n\pi t) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi t)$$

Match the coefficients of $\sin(n\pi t)$ for each $n = 1, 2, \ldots$

$$(64 - n^2\pi^2) B_n = \frac{2(-1)^{n+1}}{n\pi}$$

$$\Rightarrow B_n = \frac{2(-1)^{n+1}}{n\pi(64 - n^2\pi^2)}$$

* Note, since $64 - n^2\pi^2 \neq 0$, each B_n is well defined.

$$\boxed{\text{The particular solution } \quad x_p = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi(64 - n^2\pi^2)} \sin(n\pi t)}$$