November 19 Math 3260 sec. 53 Fall 2025

Chapter 6 Eigenstuff

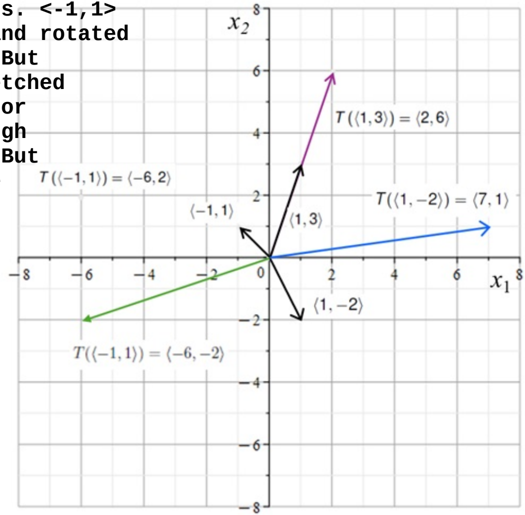
Consider the linear transformation $T : R^2 \to R^2$ defined by $T(\vec{x}) = A\vec{x}$ with $A = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}$.

1. Evaluate $T(\langle -1, 1\rangle)$ $= \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \langle -1, 1\rangle = \langle -6, -2\rangle$

2. Evaluate $T(\langle 1, 3\rangle)$ $= \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \langle 1, 3\rangle = \langle 2, 6\rangle$

3. Evaluate $T(\langle 1, -2\rangle)$ $= \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \langle 1, -2\rangle = \langle 7, 1$

Note that T seems to rotate and stretch the input vectors. <-1,1> gets stretched and rotated as does <1,-2>. But they aren't stretched by the same factor or rotated through the same angle. But T doesn't rotate the input <1,3> at all. It just scales it by a factor of 2.



$T(\langle 1,3\rangle)$

$= \langle 2,6\rangle$ is

in

Span$\{\langle 1,3\rangle\}$.

T(<1,3>)= 2<1,3>

Figure: Plot of standard representations of $\langle -1,1\rangle$, $T(\langle -1,1\rangle)$, $\langle 1,3\rangle$, $T(\langle 1,3\rangle)$, $\langle 1,-2\rangle$, and $T(\langle 1,-2\rangle)$ together.

Example Continued...: $A = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}$

Show that there is a special set of nonzero vectors in R^2 with the property $A\vec{x} = 2\vec{x}$.

Let $\vec{x} = \langle x_1, x_2 \rangle$.

$$A\vec{x} = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \langle x_1, x_2 \rangle = \langle 5x_1 - x_2, 3x_1 + x_2 \rangle$$

$$2\vec{x} = \langle 2x_1, 2x_2 \rangle \quad \text{set} \quad \langle 5x_1 - x_2, 3x_1 + x_2 \rangle = \langle 2x_1, 2x_2 \rangle$$

$$\Rightarrow \begin{array}{l} 5x_1 - x_2 = 2x_1 \\ 3x_1 + x_2 = 2x_2 \end{array} \Rightarrow \begin{array}{l} 5x_1 - 2x_1 - x_2 = 0 \\ 3x_1 + x_2 - 2x_2 = 0 \end{array}$$

$$\Rightarrow \begin{array}{l} 3x_1 - x_2 = 0 \\ 3x_1 - x_2 = 0 \end{array} \Rightarrow \begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix} \langle x_1, x_2 \rangle = \langle 0, 0 \rangle$$

$$\left[\begin{array}{cc|c} 3 & -1 & 0 \\ 3 & -1 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{l} x_1 = \frac{1}{3}x_2 \\ x_2 - \text{free} \end{array}$$

solutions $\vec{x} = \langle \frac{1}{3}x_2, x_2 \rangle = x_2 \langle \frac{1}{3}, 1 \rangle$

Every vector in $\text{Span}\{\langle \frac{1}{3}, 1\rangle\}$

satisfies $A\vec{x} = 2\vec{x}$.

* If $x_2 = 3$, we get $\vec{x} = \langle 1, 3 \rangle$.

Note that A doesn't scale every vector by 2, only the ones in Span{<1/3, 1>}.For these special vectors, the matrix-vector product ends up being a scalar-vector product, specifically with the scalar 2.

What are Eigen *things*?

We'll focus on linear transformations $T : R^n \to R^n$ with square matrix A such that $T(\vec{x}) = A\vec{x}$, and consider equations of the form

$$A\vec{x} = \lambda\vec{x}$$

with λ a scalar and $\vec{x}$ a nonzero vector. We'll call scalars like λ **eigenvalues** and vectors like $\vec{x}$ **eigenvectors**. We'll also consider things like **eigenspaces** and **eigenbases**.

Questions:

- Does a matrix have such scalars and vectors?
- How would we find them?
- What do the tell us about a matrix?
- What can we do with them?

Note that the left is a matrix-vector, and the right a scalar-vector. But both sides are vectors in R^n, so the equation is plausible.

6.1 The Determinant

The determinant is a scalar valued function on $M_{n\times n}$. The determinant is related to various properties of a matrix, most notably invertibility.

2×2 Determinant

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The **determinant** of A, denoted $\det(A)$, is the number

$$\det(A) = ad - bc.$$

Remark: 1×1 matrices have limited usefulness. We will define the determinant of the 1×1 matrix $A = [a]$ to be $\det(A) = a$.

Example: Evaluate the determinant of each matrix.

$$A = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix}$$

$\det(A) = 5(1) - 3(-1) = 8$

$\det(B) = 3(-1) - 3(-1) = 0$

Submatrices

Let $A = [a_{ij}]$ be an $n \times n$ matrix. The notation

$$A_{ij}$$

will denote the $(n-1)\times(n-1)$ matrix obtained from A by removing the i^{th} row and the j^{th} column.

For example, if $A = [a_{ij}]$ is a 3×3 matrix, we can form nine different 2×2 matrices A_{ij}.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cancel{a_{13}} \\ \cancel{a_{21}} & \cancel{a_{22}} & \cancel{a_{23}} \\ a_{31} & a_{32} & \cancel{a_{33}} \end{bmatrix} \quad \to \quad A_{23} = \begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{bmatrix}.$$

$$A = \begin{bmatrix} \cancel{a_{11}} & a_{12} & a_{13} \\ \cancel{a_{21}} & a_{22} & a_{23} \\ \cancel{a_{31}} & \cancel{a_{32}} & \cancel{a_{33}} \end{bmatrix} \quad \to \quad A_{31} = \begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}$$

Minors & Cofactors

Let A be an $n \times n$ matrix, $n \geq 2$. The ijth **minor** of A is the determinant of the $(n-1) \times (n-1)$ matrix A_{ij}. That is, $\det(A_{ij})$ is the ijth minor of A.

a minor is a scalar

Let A be an $n \times n$ matrix, $n \geq 2$.

$$ij\text{th } \textbf{cofactor} \text{ of } A \ = \ (-1)^{i+j} \det(A_{ij}).$$

Cofactor Signs

The factor $(-1)^{i+j}$ gives an alternating sign based on the position of a_{ij} in the matrix.

$$\begin{bmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

$n \times n$ Determinant

Let $A = \left[a_{ij}\right]$ be an $n \times n$ matrix. The determinant of A, denoted $\det(A)$ is given by

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(A_{1j}). \tag{1}$$

The sum in equation (1) is called a **cofactor expansion** across the first row of A.

Remark: Note what this sum is. Take each entries in the first row, a_{1j}, multiply it by its corresponding cofactor $(-1)^{1+j}\det(A_{1j})$, and then add them all together:

$$\begin{array}{rcccccccc} \det(A) & = & (-1)^2 a_{11}\det(A_{11}) & + & (-1)^3 a_{12}\det(A_{12}) & + & \cdots & + & (-1)^{n+1} a_{1n}\det(A_{1n}) \\ & = & a_{11}\det(A_{11}) & - & a_{12}\det(A_{12}) & + & \cdots & + & (-1)^{n+1} a_{1n}\det(A_{1n}) \end{array}$$

Example

$\det(A) = +a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + a_{13}\det(A_{13})$

Find $\det(A)$ if $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 2 \\ 1 & -1 & 0 \end{bmatrix}$.

$A_{11} = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$ $\quad A_{12} = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$ $\quad A_{13} = \begin{bmatrix} 0 & 3 \\ 1 & -1 \end{bmatrix}$

$\det(A_{11}) = 3(0) - (-1)(2)$
$= 2$

$\det(A_{12}) = 0^2 - 1(2)$
$= -2$

$\det(A_{13}) = 0(-1) - (1)(3)$
$= -3$

$\det(A) = 1(2) - 2(-2) + (-1)(-3)$
$= 2 + 4 + 3 = 9$

Alternative Cofactor Expansions

We can actually take a determinant by a cofactor expansion across any row. Fix a number for i (between 1 and n).

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}). \tag{2}$$

Similarly, we can compute $\det(A)$ using a cofactor expansion down any columns. Fix a number j (between 1 and n).

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}). \tag{3}$$

Remark: Formulas (2) and (3) look almost identical. But in formula (2), the value if i doesn't change. In the formula (3), the value if j doesn't change.

Example

$\det(A) = 9$

Find $\det(A)$ if $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 2 \\ 1 & -1 & 0 \end{bmatrix}$ by cofactor expansion across the second row and down the third column.

$$\det(A) = -a_{21}\det(A_{21}) + a_{22}\det(A_{22}) - a_{23}\det(A_{23})$$

$$= -0\det\begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix} + 3\det\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} - 2\det\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$= 0 + 3\big(1\cdot 0 - (1)(-1)\big) - 2\big((1)(-1) - 1(2)\big)$$

$$= 0 + 3(1) - 2(-1-2)$$

$$= 0 + 3 + 6 = 9$$

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 2 \\ 1 & -1 & 0 \end{bmatrix}$$

$$\det(A) = +\, a_{13} \det(A_{13}) - a_{23} \det(A_{23}) + a_{33} \det(A_{33})$$

$$= +(-1) \det\begin{bmatrix} 0 & 3 \\ 1 & -1 \end{bmatrix} - 2 \det\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} + 0 \det\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

$$= -1\,(0-3) - 2\,(-1-2) + 0$$

$$= 3 + 6 \quad = 9$$

Note that the negative sign is because the entry is -1. The cofactor sign starts with + but the actual number a_{13} is negative.

Properties of the Determinant

Evaluate $\det(A)$ if $A = \begin{bmatrix} 2 & 0 & -1 & 4 \\ 4 & 0 & 5 & -3 \\ -6 & 0 & 1 & 1 \\ 2 & 0 & 13 & -1 \end{bmatrix}$.

Down Column 2

$$\det(A) = -0\det(A_{12}) + 0\det(A_{22}) - 0\det(A_{32}) + 0\det(A_{42})$$

$$= 0$$

We don't have to actually compute any cofactors since each entry in the column is zero. If we did do the expansion across any row or down any other column, we'd still get zero as the final result.

Triangular Matrices

A matrix $A = [a_{ij}]$ is called **upper triangular** if $a_{ij} = 0$ for all $i > j$, and it's called **lower triangular** if $a_{ij} = 0$ for all $i < j$. As the names suggest, upper triangular matrices have all their nonzero entries on or above the main diagonal, and lower triangular matrices have all their nonzero entries on or below the main diagonal.

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}}_{\text{upper triangular}} \quad \underbrace{\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}}_{\text{lower triangular}}$$

A matrix that is both upper triangular and lower triangular is called a **diagonal** matrix.

$$\underbrace{\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}}_{\text{diagonal matrix}}$$

Since a cofactor expansion is the same across any row or down any column...

Theorem

Let A be an $n \times n$ matrix.

1. If $\vec{0}_n$ is a row vector or a column vector of A, then $\det(A) = 0$.
2. $\det(A^T) = \det(A)$.
3. If A is a triangular matrix (upper, lower or diagonal), the $\det(A)$ is the product of the diagonal entries

$$\det(A) = a_{11}a_{22}\cdots a_{nn}$$