

Section 16: Laplace Transforms of Derivatives and IVPs

For $y = y(t)$ defined on $[0, \infty)$ having derivatives y' , y'' and so forth, if

$$\mathcal{L}\{y(t)\} = Y(s),$$

then

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0),$$

$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0),$$

:

:

$$\mathcal{L}\left\{\frac{d^n y}{dt^n}\right\} = s^n Y(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - y^{(n-1)}(0).$$

Solving IVPs

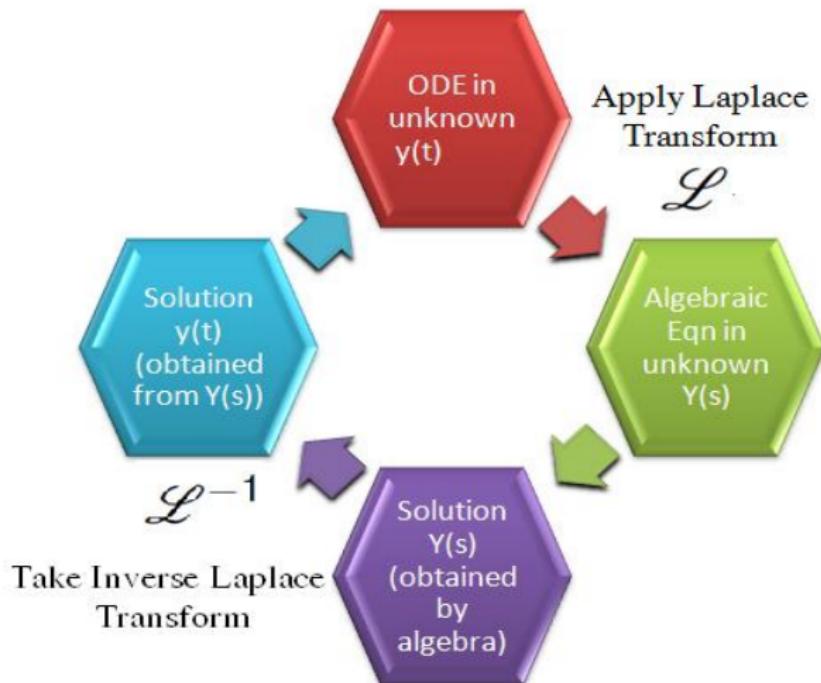
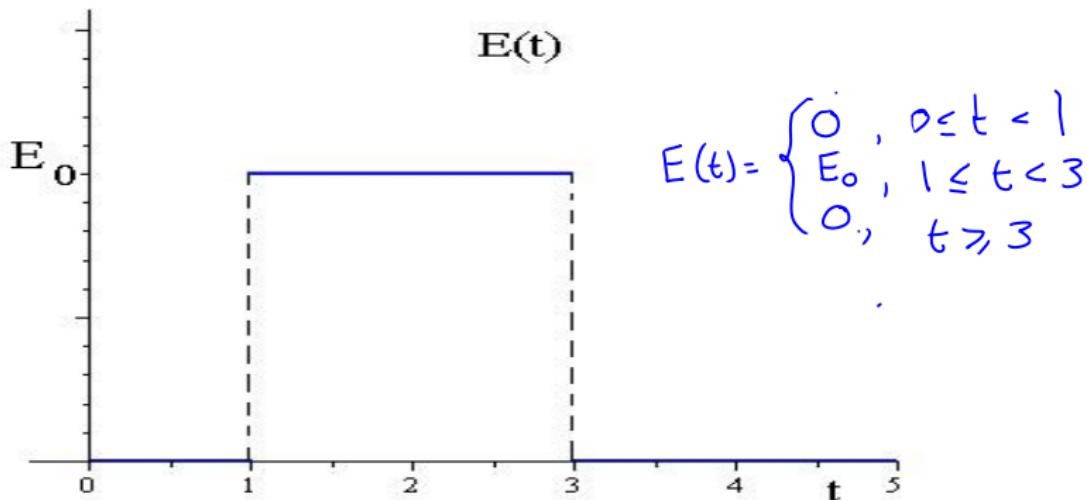


Figure: We use the Laplace transform to turn our DE into an algebraic equation. Solve this transformed equation, and then transform back.

Solve the IVP

An LR-series circuit has inductance $L = 1\text{h}$, resistance $R = 10\Omega$, and applied force $E(t)$ whose graph is given below. If the initial current $i(0) = 0$, find the current $i(t)$ in the circuit.

$$Li' + Ri = E$$



$$E(t) = 0 - 0 \cdot u(t-1) + E_0 \cdot u(t-1) - E_0 \cdot u(t-3) + 0 \cdot u(t-3)$$

LR Circuit Example $L = 1$, $R = 10$

$$\frac{di}{dt} + 10i = E_0 u(t-1) - E_0 u(t-3), \quad i(0) = 0$$

Let $I(s) = \mathcal{L}\{i(t)\}$.

$$\mathcal{L}\{i' + 10i\} = \mathcal{L}\{E_0 u(t-1) - E_0 u(t-3)\},$$

$$\mathcal{L}\{i'\} + 10\mathcal{L}\{i\} = E_0 \mathcal{L}\{u(t-1)\} - E_0 \mathcal{L}\{u(t-3)\}.$$

$$sI(s) - i(0) + 10I(s) = E_0 \frac{e^{-s}}{s} - E_0 \frac{e^{-3s}}{s}$$

$$i(0) = 0$$

$$(s+10) I(s) = E_0 \frac{e^{-s}}{s} - E_0 \frac{e^{-3s}}{s}$$

$$I(s) = E_0 \frac{e^{-s}}{s(s+10)} - E_0 \frac{e^{-3s}}{s(s+10)}$$

We need a decomposition of $\frac{1}{s(s+10)}$

$$\frac{1}{s(s+10)} = \frac{A}{s} + \frac{B}{s+10}$$

Clear fractions.

$$1 = A(s+10) + Bs$$

$$\text{set } s=0 \quad 1=10A \quad A=\frac{1}{10}$$

$$s=-10 \quad 1=-10B \quad B=-\frac{1}{10}$$

$$I(s) = E_0 \bar{e}^s \left(\frac{\frac{1}{10}}{s} - \frac{\frac{1}{10}}{s+10} \right) - E_0 \bar{e}^{-3s} \left(\frac{\frac{1}{10}}{s} - \frac{\frac{1}{10}}{s+10} \right)$$

$$\mathcal{L}^{-1}\{e^{-as} F(s)\} = f(t-a) u(t-a)$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\}$$

we need

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\left\{ \frac{E_0}{s} - \frac{E_0}{s+10} \right\} \\ &= \frac{E_0}{10} \mathcal{L}^{-1}\left(\frac{1}{s}\right) - \frac{E_0}{10} \mathcal{L}^{-1}\left(\frac{1}{s+10}\right) \\ &= \frac{E_0}{10} - \frac{E_0}{10} e^{-10t} \end{aligned}$$

$$I(s) = E_0 \bar{e}^s \left(\frac{\frac{1}{10}}{s} - \frac{\frac{1}{10}}{s+10} \right) - E_0 \bar{e}^{-3s} \left(\frac{\frac{1}{10}}{s} - \frac{\frac{1}{10}}{s+10} \right)$$

The current $i(t) = \mathcal{L}^{-1}\{I(s)\}$

$$i(t) = \left(\frac{E_0}{10} - \frac{E_0}{10} e^{-10(t-1)} \right) u(t-1) - \left(\frac{E_0}{10} - \frac{E_0}{10} e^{-10(t-3)} \right) u(t-3)$$

We can express this piecewise defined function in stacked notation

$$i(t) = \begin{cases} 0, & 0 \leq t < 1 \\ \frac{E_0}{10} - \frac{E_0}{10} e^{-10(t-1)}, & 1 \leq t < 3 \\ -\frac{E_0}{10} e^{-10(t-1)} + \frac{E_0}{10} e^{-10(t-3)}, & t \geq 3 \end{cases}$$

Example

A 1 kg mass is attached to a spring with a spring constant of 10 N/m. A dashpot provides damping numerically equal to 2 times the instantaneous velocity. At time $t = 0$ and again at time $t = 2$ seconds, a unit impulse is applied. If the mass starts from rest at equilibrium, determine the displacement, $x(t)$, of the mass for all $t > 0$.

To do this, use the method of Laplace transforms to solve the IVP

$$x'' + 2x' + 10x = \delta(t - 0) + \delta(t - 2), \quad x(0) = 0, \quad x'(0) = 0.$$

Recall that $\mathcal{L}\{\delta(t - a)\} = e^{-as}$.

Notice that this ODE has the form $mx'' + \beta x' + kx = f(t)$. The applied impulse at $t = a$ seconds is modeled by $f(t) = \delta(t - a)$.

$$x'' + 2x' + 10x = \delta(t - 0) + \delta(t - 2), \quad x(0) = 0, \quad x'(0) = 0.$$

$$\mathcal{L}\{x(t)\} = X(s)$$

$$\mathcal{L}\{x'' + 2x' + 10x\} = \mathcal{L}\{\delta(t) + \delta(t-2)\}$$

$$\mathcal{L}\{x''\} + 2\mathcal{L}\{x'\} + 10\mathcal{L}\{x\} = \mathcal{L}\{\delta(t)\} + \mathcal{L}\{\delta(t-2)\}$$

$$s^2X(s) - s\cancel{x(0)} - \cancel{x'(0)} + 2 \left(sX(s) - \cancel{x(0)} \right) + 10X(s) = e^0 + e^{-2s}$$

$$(s^2 + 2s + 10)X(s) = 1 + e^{-2s}$$

$$X(s) = \frac{1}{s^2 + 2s + 10} + \frac{e^{-2s}}{s^2 + 2s + 10}$$

$s^2 + 2s + 10$ is irreducible, complete the square.

$$s^2 + 2s + 1 - 1 + 10 = (s+1)^2 + 9$$

$$X(s) = \frac{1}{(s+1)^2 + 9} + \frac{e^{-2s}}{(s+1)^2 + 9}$$

Need $f(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2 + 3^2}\right\}$

$$= e^{-t} \mathcal{L}^{-1}\left\{\frac{1}{s^2 + 3^2}\right\}$$

$$= \frac{1}{3} e^{-t} \mathcal{L}^{-1}\left\{\frac{3}{s^2 + 3^2}\right\} = \frac{1}{3} e^{-t} \sin(3t)$$

$$X(s) = \frac{1}{(s+1)^2 + 9} + \frac{e^{-2s}}{(s+1)^2 + 9}$$

The displacement $X(t) = \mathcal{L}^{-1}\{X(s)\}$

$$X(t) = \frac{1}{3} e^t \sin(3t) + \frac{1}{3} e^{-(t-2)} \sin(3(t-2)) u(t-2)$$