November 1 Math 2306 sec. 51 Fall 2024

Section 13: The Laplace Transform

Consider and IVP with piecewise forcing,

$$Lq'' + Rq' + \frac{q}{C} = \begin{cases} E_0, & 0 < t < \epsilon \\ 0, & t \geq \epsilon \end{cases} \qquad q(0) = 0, \quad i(0) = 0$$

$$mx'' + bx' + kx = \begin{cases} 0, & 0 < t < t_1 \\ a\cos(\gamma t), & t_1 < t < t_2 \\ 0, & t > t_2 \end{cases} \qquad x(0) = x_0, \quad x'(0) = v_0$$

Remark: We can solve problems like this with our present tools by solving multiple IVPs along with a continuity argument. Laplace transforms will provide a new solution method that allows us to solve the whole problem in a single process.

Section 13: The Laplace Transform

A quick word about functions of 2-variables:

Suppose $G(s,t)$ is a function of two independent variables (s and t) defined over some rectangle in the plane $a \le t \le b$, $c \le s \le d$. If we compute an integral with respect to one of these variables, say t,

$$\int_{\alpha}^{\beta} G(s,t)\,dt$$

- the result is a function of the remaining variable s, and
- the variable s is treated as a constant while integrating with respect to t.

For Example...

Assume that $s \neq 0$ and $b > 0$. Compute the integral

$$\int_0^b e^{-st}\,dt = \frac{1}{-s} e^{-st} \Big|_0^b$$

$$= \frac{-1}{s}\left(e^{-sb} - e^{-s(0)}\right)$$

$$= \frac{-1}{s}\left(e^{-sb} - 1\right) = \frac{1 - e^{-sb}}{s}$$

Integral Transform

An **integral transform**[a] is a mapping that assigns to a function $f(t)$ another function $F(s)$ via an integral of the form

$$\int_a^b K(s,t) f(t)\, dt.$$

- The function K is called the **kernel** of the transformation.
- The limits a and b may be finite or infinite.
- The integral may be improper so that convergence/divergence must be considered.
- This transform is **linear** in the sense that

$$\int_a^b K(s,t)(\alpha f(t) + \beta g(t))\, dt = \alpha \int_a^b K(s,t) f(t)\, dt + \beta \int_a^b K(s,t) g(t)\, dt.$$

[a] More precisely, this is the definition of a **linear** integral transform.

The Laplace Transform

Definition: The Laplace Transform

Let $f(t)$ be piecewise continuous on $[0, \infty)$. The Laplace transform of f, denoted $\mathscr{L}\{f(t)\}$ is given by.

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt. \quad = F(s)$$

We will often use the upper case/lower case convention that $\mathscr{L}\{f(t)\}$ will be represented by $F(s)$. The domain of the transformation $F(s)$ is the set of all s such that the integral is convergent.

Remark 1: The **kernel** for the Laplace transform is $K(s,t) = e^{-st}$.

Remark 2: In general, s is considered a complex variable. We will generally take s to be real, but this will not restrict our use of the Laplace transform.

Limits at Infinity e^{-st}

If $s > 0$, evaluate

$$\lim_{t\to\infty} e^{-st} = 0$$

If $s>0$, then $-st < 0$, $-st \to -\infty$ as $t \to \infty$

If $s < 0$, evaluate

$$\lim_{t\to\infty} e^{-st} = \infty$$

If $s<0$, then $-st > 0$, $-st \to +\infty$ as $t \to \infty$

Find[1] the Laplace transform of $f(t) = 1$.

By definition, $\mathcal{L}\{1\} = \int_0^\infty e^{-st}\,1\,dt = \int_0^\infty e^{-st}\,dt$.

If $s=0$, the integral is $\int_0^\infty dt$

$$\int_0^\infty dt = \lim_{b\to\infty} \int_0^b dt = \lim_{b\to\infty} t\Big|_0^b = \lim_{b\to\infty} (b-0) = \infty$$

The integral diverges $\Rightarrow$ zero is not in the domain of $\mathcal{L}\{1\}$.

For $s \neq 0$, we have $\int_0^\infty e^{-st}\,dt =$

[1] Unless stated otherwise, the domain for each example is $[0, \infty)$. That is, f is defined for $0 \le t < \infty$.

$$= \lim_{b \to \infty} \int_0^b e^{-st}\, dt = \lim_{b \to \infty} \frac{1 - e^{-sb}}{s}$$

Convergence requires $s > 0$.

when $s > 0$, $\mathcal{L}\{1\} = \lim_{b \to \infty} \frac{1 - e^{-sb}}{s} = \frac{1}{s}$.

So, $\mathcal{L}\{1\} = \frac{1}{s}$ with $s > 0$.

Find the Laplace transform of $f(t) = t$.

By definition, $\mathcal{L}\{t\} = \int_0^\infty e^{-st}\, t\, dt$

It's easy to show that the integral diverges if $s=0$. For $s \neq 0$,

Int by parts

$u = t \quad du = dt$

$v = \frac{-e^{-st}}{s} \quad dv = e^{-st}\, dv$

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st}\, t\, dt$$

$$= \frac{-t}{s} e^{-st} \Big|_0^\infty - \int_0^\infty \frac{-1}{s} e^{-st}\, dt$$

Convergence requires $s > 0$.

For $s>0$

$$\mathcal{L}\{t\} = 0 - 0 + \frac{1}{s}\int_0^\infty e^{-st}\,dt$$

$$= \frac{1}{s}\int_0^\infty e^{-st}\,dt$$

$$= \frac{1}{s}\mathcal{L}\{1\} = \frac{1}{s}\left(\frac{1}{s}\right) = \frac{1}{s^2}$$

$$\mathcal{L}\{t\} = \frac{1}{s^2}, \quad s>0.$$

A piecewise defined function

Find the Laplace transform of f defined by

$$f(t) = \begin{cases} 2t, & 0 \le t < 10 \\ 0, & t \ge 10 \end{cases}$$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^{10} e^{-st} f(t)\,dt + \int_{10}^\infty e^{-st} f(t)\,dt$$

$$= \int_0^{10} e^{-st} (2t)\,dt + \int_{10}^\infty e^{-st} (0)\,dt$$

$$= \int_0^{10} 2t\, e^{-st}\,dt$$

We'll finish this next time. Note that in this case, the integral isn't going to be improper.