November 1 Math 2306 sec. 54 Fall 2021

Section 16: Laplace Transforms of Derivatives and IVPs

For $y = y(t)$ defined on $[0, \infty)$ having derivatives y', y'' and so forth, if

$$\mathscr{L}\left\{y(t)\right\} = Y(s),$$

then

$$\begin{aligned}
\mathscr{L}\left\{\frac{dy}{dt}\right\} &= sY(s) - y(0),\\
\mathscr{L}\left\{\frac{d^2y}{dt^2}\right\} &= s^2Y(s) - sy(0) - y'(0),\\
&\vdots \qquad\qquad \vdots\\
\mathscr{L}\left\{\frac{d^ny}{dt^n}\right\} &= s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - y^{(n-1)}(0).
\end{aligned}$$

Solving IVPs

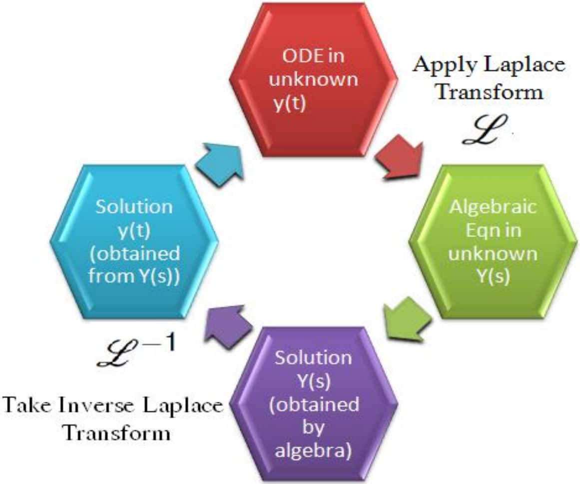
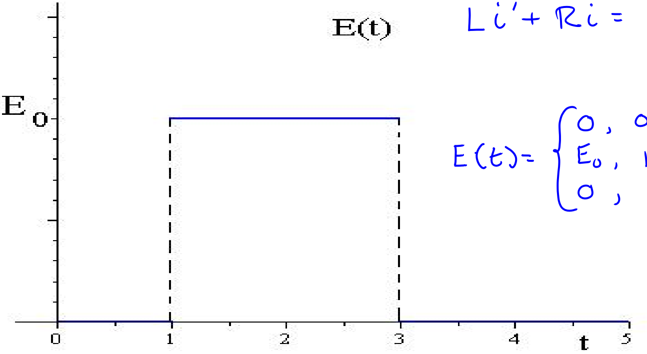


Figure: We use the Laplace transform to turn our DE into an algebraic equation. Solve this transformed equation, and then transform back.

Solve the IVP

An LR-series circuit has inductance $L = 1$h, resistance $R = 10\Omega$, and applied force $E(t)$ whose graph is given below. If the initial current $i(0) = 0$, find the current $i(t)$ in the circuit.



$Li' + Ri = E$

$$E(t) = \begin{cases} 0, & 0 \le t < 1 \\ E_0, & 1 \le t < 3 \\ 0, & t \ge 3 \end{cases}$$

$E(t) = 0 - 0\mathcal{U}(t-1) + E_0\mathcal{U}(t-1) - E_0\mathcal{U}(t-3) + 0\mathcal{U}(t-3)$

LR Circuit Example

$L=1, \quad R=10$

$$i' + 10i = E_0\,\mathcal{U}(t-1) - E_0\,\mathcal{U}(t-3), \quad i(0)=0$$

Let $\mathcal{L}\{i(t)\} = I(s)$

$$\mathcal{L}\{i' + 10i\} = \mathcal{L}\{E_0\,\mathcal{U}(t-1) - E_0\,\mathcal{U}(t-3)\}$$

$$\mathcal{L}\{i'\} + 10\,\mathcal{L}\{i\} = E_0\,\mathcal{L}\{\mathcal{U}(t-1)\} - E_0\,\mathcal{L}\{\mathcal{U}(t-3)\}$$

$$sI(s) - \underbrace{\cancel{i(0)}}_{0} + 10\,I(s) = E_0\frac{e^{-s}}{s} - E_0\frac{e^{-3s}}{s}$$

$i(0)=0$

$$(s+10)\,I(s) = \frac{E_0\,e^{-s}}{s} - \frac{E_0\,e^{-3s}}{s}$$

$$I(s) = \frac{E_0 e^{-s}}{s(s+10)} - \frac{E_0 e^{-3s}}{s(s+10)}.$$

We need to decompose $\dfrac{1}{s(s+10)}$.

$$\frac{1}{s(s+10)} = \frac{A}{s} + \frac{B}{s+10}$$ clear fractions

$$1 = A(s+10) + Bs$$

Set $s=0$, $\quad 1 = 10A \implies A = \frac{1}{10}$

$s=-10 \quad 1 = -10B \implies B = \frac{-1}{10}$

$$\frac{1}{s(s+10)} = \frac{\frac{1}{10}}{s} - \frac{\frac{1}{10}}{s+10}$$

$$I(s) = E_0 e^{-s}\left(\frac{\frac{1}{10}}{s} - \frac{\frac{1}{10}}{s+10}\right) - E_0 e^{-3s}\left(\frac{\frac{1}{10}}{s} - \frac{\frac{1}{10}}{s+10}\right)$$

Recall $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathcal{U}(t-a)$

where $f(t) = \mathcal{L}^{-1}\{F(s)\}$.

we need

$$f(t) = \mathcal{L}^{-1}\left\{\frac{E_0}{10}\left(\frac{1}{s} - \frac{1}{s+10}\right)\right\}$$

$$= \frac{E_0}{10}\left(\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+10}\right\}\right)$$

$$f(t) = \frac{E_0}{10} - \frac{E_0}{10} e^{-10t}$$

$$I(s) = E_0 e^{-s}\left(\frac{\frac{1}{10}}{s} - \frac{\frac{1}{10}}{s+10}\right) - E_0 e^{-3s}\left(\frac{\frac{1}{10}}{s} - \frac{\frac{1}{10}}{s+10}\right)$$

The current $i(t) = \mathscr{L}^{-1}\{I(s)\}$

$$\boxed{i(t) = \left(\frac{E_0}{10} - \frac{E_0}{10} e^{-10(t-1)}\right)\mathscr{U}(t-1) - \left(\frac{E_0}{10} - \frac{E_0}{10} e^{-10(t-3)}\right)\mathscr{U}(t-3)}$$

We can write $i(t)$ using the stacked notation for a piecewise defined function

$$i(t) = \begin{cases} 0, & 0 \le t < 1 \\ \frac{E_0}{10} - \frac{E_0}{10} e^{-10(t-1)}, & 1 \le t < 3 \\ \frac{E_0}{10} e^{-10(t-3)} - \frac{E_0}{10} e^{-10(t-1)}, & t \geq 3 \end{cases}$$

$$\frac{E_0}{10} - \frac{E_0}{10} e^{-10(t-1)} - \frac{E_0}{10} + \frac{E_0}{10} e^{-10(t-3)}$$

Example

A 1 kg mass is attached to a spring with a spring constant of 10 N/m. A dashpot provides damping numerically equal to 2 times the instantaneous velocity. At time $t = 0$ and again at time $t = 2$ seconds, a unit impulse is applied. If the mass starts from rest at equilibrium, determine the displacement, $x(t)$, of the mass for all $t > 0$.

To do this, use the method of Laplace transforms to solve the IVP

$$x'' + 2x' + 10x = \delta(t-0) + \delta(t-2), \quad x(0) = 0, \quad x'(0) = 0.$$

Recall that $\mathscr{L}\{\delta(t-a)\} = e^{-as}$.

Notice that this ODE has the form $mx'' + \beta x' + kx = f(t)$. The applied impulse at $t = a$ seconds is modeled by $f(t) = \delta(t-a)$.

$$x'' + 2x' + 10x = \delta(t-0) + \delta(t-2), \quad x(0) = 0, \quad x'(0) = 0.$$

Let $\mathcal{L}\{x\} = X(s)$

$$\mathcal{L}\{x'' + 2x' + 10x\} = \mathcal{L}\{\delta(t) + \delta(t-2)\}.$$

$$\mathcal{L}\{x''\} + 2\mathcal{L}\{x'\} + 10\mathcal{L}\{x\} = \mathcal{L}\{\delta(t)\} + \mathcal{L}\{\delta(t-2)\}$$

$$s^2X(s) - s\underbrace{x(0)}_{0} - \underbrace{x'(0)}_{0} + 2\left(sX(s) - \underbrace{x(0)}_{0}\right) + 10X(s) = 1 + e^{-2s}$$

$$(s^2 + 2s + 10)X(s) = 1 + e^{-2s}$$

$$X(s) = \frac{1}{s^2+2s+10} + \frac{e^{-2s}}{s^2+2s+10}$$

$s^2+2s+10$ is irreducible, complete the square.

$$s^2+2s+1-1+10 = (s+1)^2+9$$

Hence

$$X(s) = \frac{1}{(s+1)^2+9} + \frac{e^{-2s}}{(s+1)^2+9}$$

we need

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+9}\right\}$$

$$= e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+3^2}\right\}$$

$$= \frac{1}{3}e^{-t}\mathcal{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\}$$

$$= \frac{1}{3}e^{-t}\sin(3t)$$

$$X(s) = \frac{1}{(s+1)^2+9} + \frac{e^{-2s}}{(s+1)^2+9}$$

$$\boxed{\begin{array}{l}\text{The displacement } x(t) = \mathcal{L}^{-1}\{X(s)\} \\ x(t) = \frac{1}{3}e^{-t}\sin(3t) + \frac{1}{3}e^{-(t-2)}\sin(3(t-2))\mathcal{U}(t-2)\end{array}}$$