November 20 Math 2306 sec. 53 Fall 2024

Section 16: Laplace Transforms of Derivatives and IVPs
Systems of IVPs

We can solve a system of ODEs using Laplace transforms. Here, we'll consider systems that are

- linear,
- having initial conditions at $t = 0$, and
- constant coefficient.

Let's see it in action (i.e. with a couple of examples).

Cramer's Rule

We can solve a linear system using substitution or elimination. Cramer's rule is quick for small (e.g., 2×2 or 3×3) square systems.

$$\begin{array}{ccccc} ax & + & by & = & e \\ cx & + & dy & = & f \end{array}$$

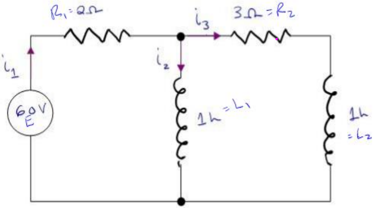
Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be the coefficient matrix, and define A_x and A_y to be the matrices obtained from A by replacing the first, respectively second, column with the right hand sides of the equations.

$$A_x = \begin{bmatrix} e & b \\ f & d \end{bmatrix}, \quad \text{and} \quad A_y = \begin{bmatrix} a & e \\ c & f \end{bmatrix}.$$

The solution of the system can be stated in terms of ratios of determinants.

$$x = \frac{\det(A_x)}{\det(A)} \quad \text{and} \quad y = \frac{\det(A_y)}{\det(A)}.$$

Example



$$
\begin{aligned}
L_1 i_2' &+ R_1(i_2+i_3) & &= E \\
L_2 i_3' &+ R_1(i_2+i_3) &+ R_2 i_3 &= E
\end{aligned}
$$

Figure: If we label current i_2 as $x(t)$ and current i_3 as $y(t)$, we get the system of equations below. (Assuming $i_1(0) = 0$.)

Solve the system of equations

$$
\begin{aligned}
\frac{dx}{dt} &= -2x - 2y + 60, \quad x(0) = 0 \\
\frac{dy}{dt} &= -2x - 5y + 60, \quad y(0) = 0
\end{aligned}
$$

Let $X(s) = \mathcal{L}\{x(t)\}$ and $Y(s) = \mathcal{L}\{y(t)\}$.

$$\frac{dx}{dt} = -2x - 2y + 60, \quad x(0) = 0$$
$$\frac{dy}{dt} = -2x - 5y + 60, \quad y(0) = 0$$

$$\mathcal{L}\{x'\} = \mathcal{L}\{-2x - 2y + 60\}$$
$$= -2\mathcal{L}\{x\} - 2\mathcal{L}\{y\} + 60\mathcal{L}\{1\}$$

$$sX(s) - \underbrace{x(0)}_{0=} = -2X(s) - 2Y(s) + \frac{60}{s}$$

$$\mathcal{L}\{y'\} = \mathcal{L}\{-2x - 5y + 60\}$$
$$= -2\mathcal{L}\{x\} - 5\mathcal{L}\{y\} + 60\mathcal{L}\{1\}$$

$$sY(s) - \overbrace{y(0)}^{=0} = -2X(s) - 5Y(s) + \frac{60}{s}.$$

$$sX = -2X - 2Y + \frac{60}{s}$$

$$sY = -2X - 5Y + \frac{60}{s}$$

$$(s+2)X + 2Y = \frac{60}{s}$$

$$2X + (s+5)Y = \frac{60}{s}$$

we'll use Cramer's rule

In matrix format

$$\begin{bmatrix} s+2 & 2 \\ 2 & s+5 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \frac{60}{s} \\ \frac{60}{s} \end{bmatrix}$$

$$A = \begin{bmatrix} s+2 & 2 \\ 2 & s+5 \end{bmatrix}, \quad A = \begin{bmatrix} \frac{60}{s} & 2 \\ \frac{60}{s} & s+5 \end{bmatrix}, \quad A_Y = \begin{bmatrix} s+2 & \frac{60}{s} \\ 2 & \frac{60}{s} \end{bmatrix}$$

$$\det(A) = (s+2)(s+5) - 2\cdot 2 = s^2 + 7s + 10 - 4 = s^2 + 7s + 6$$

$$\det(A_x) = \frac{60}{s}(s+5) - \frac{60}{s}(2) = \frac{60}{s}(s+5-2) = \frac{60}{s}(s+3)$$

$$\det(A_y) = (s+2)\frac{60}{s} - 2\left(\frac{60}{s}\right) = \frac{60}{s}(s+2-2) = \frac{60}{s}(s) = 60$$

$$X(s) = \frac{\det(A_x)}{\det(A)} = \frac{\frac{60}{s}(s+3)}{s^2+7s+6} = \frac{60(s+3)}{s(s+1)(s+6)}$$

$$Y(s) = \frac{\det(A_y)}{\det(A)} = \frac{60}{s^2+7s+6} = \frac{60}{(s+1)(s+6)}$$

we'll do PFD on both ratios.

$$X(s) = \frac{60(s+3)}{s(s+1)(s+6)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+6}$$

with some effort, $A = 30$ $B = -24$ $C = -6$

$$Y(s) = \frac{60}{(s+1)(s+6)} = \frac{D}{s+1} + \frac{E}{s+6}$$

$D = 12$, $E = -12$

$$X(s) = \frac{30}{s} - \frac{24}{s+1} - \frac{6}{s+6}$$

$$Y(s) = \frac{12}{s+1} - \frac{12}{s+6}$$

The solution $x(t) = \mathcal{L}^{-1}\{X(s)\}$ and $y(t) = \mathcal{L}^{-1}\{Y(s)\}$.

$$x(t) = \mathcal{L}^{-1}\left\{\frac{30}{s} - \frac{24}{s+1} - \frac{6}{s+6}\right\}$$

$$x(t) = 30 - 24e^{-t} - 6e^{-6t}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{12}{s+1} - \frac{12}{s+6}\right\}$$

$$y(t) = 12e^{-t} - 12e^{-6t}$$

The solution to the system is

$$\boxed{\begin{aligned} x(t) &= 30 - 24e^{-t} - 6e^{-6t} \\ y(t) &= 12e^{-t} - 12e^{-6t} \end{aligned}}$$

Check the I.C.

$$x(0) = 30 - 24e^{0} - 6e^{0} = 0$$

$$y(0) = 12e^{0} - 12e^{0} = 0$$

✓

Solving A System

Solve the system of initial value problems. Assume $t_0 \geq 0$ is fixed.

$$\begin{array}{rcrcrcl} x' & - & 4x & - & y & = & \delta(t - t_0), \qquad x(0) = 0 \\ & & 2x & + & y' & = & y, \qquad\qquad\quad y(0) = 0 \end{array}$$

Let $X = \mathcal{L}\{x\}$ and $Y = \mathcal{L}\{y\}$.

$\mathcal{L}\{x' - 4x - y\} = \mathcal{L}\{\delta(t - t_0)\}$

$\mathcal{L}\{2x + y'\} = \mathcal{L}\{y\}$

$\mathcal{L}\{x'\} - 4\mathcal{L}\{x\} - \mathcal{L}\{y\} = \mathcal{L}\{\delta(t - t_0)\}$

$2\mathcal{L}\{x\} + \mathcal{L}\{y'\} = \mathcal{L}\{y\}$

$$sX - \overset{=0}{x(0)} - 4X - Y = e^{-t_0 s}$$

$$2X + sY - \underset{=0}{y(0)} = Y$$

$$(s-4)X - Y = e^{-t_0 s}$$

Cramer's Rule

$$2X + (s-1)Y = 0$$

$$\begin{bmatrix} s-4 & -1 \\ 2 & s-1 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} e^{-t_0 s} \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} s-4 & -1 \\ 2 & s-1 \end{bmatrix}, \quad A_X = \begin{bmatrix} e^{-t_0 s} & -1 \\ 0 & s-1 \end{bmatrix}, \quad A_Y = \begin{bmatrix} s-4 & e^{-t_0 s} \\ 2 & 0 \end{bmatrix}$$

$$\det(A) = (s-4)(s-1) - (-1)(2) = s^2 - 5s + 4 + 2 = s^2 - 5s + 6$$

$$\det(A_x) = e^{-t_0 s}(s-1) - 0 = e^{-t_0 s}(s-1)$$

$$\det(A_y) = 0 - 2e^{-t_0 s} = -2e^{-t_0 s}$$

$$X(s) = \frac{\det(A_x)}{\det(A)} = \frac{e^{-t_0 s}(s-1)}{s^2-5s+6} = e^{-t_0 s}\frac{s-1}{(s-2)(s-3)}$$

$$Y(s) = \frac{\det(A_y)}{\det(A)} = \frac{-2e^{-t_0 s}}{s^2-5s+6} = e^{-t_0 s}\frac{-2}{(s-2)(s-3)}$$

We need PFDs for the rational parts.

Let $F_1(s) = \frac{s-1}{(s-2)(s-3)} = \frac{A}{s-2} + \frac{B}{s-3}$ $\quad A = -1$, $B = 2$

$F_2(s) = \frac{-2}{(s-2)(s-3)} = \frac{C}{s-2} + \frac{D}{s-3}$ $\quad C = 2$, $D = -2$

$$X(s) = e^{-t_0 s}\left(\frac{-1}{s-2} + \frac{2}{s-3}\right)$$

$$Y(s) = e^{-t_0 s}\left(\frac{2}{s-2} - \frac{2}{s-3}\right)$$

Let $f_1(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-2} + \frac{2}{s-3}\right\} = -e^{2t} + 2e^{3t}$

$f_2(t) = \mathcal{L}^{-1}\left\{\frac{2}{s-2} - \frac{2}{s-3}\right\} = 2e^{2t} - 2e^{3t}$

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = f_1(t-t_0)\,u(t-t_0)$$

$$x(t) = \left(-e^{2(t-t_0)} + 2e^{3(t-t_0)}\right)u(t-t_0)$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = f_2(t-t_0)\,u(t-t_0)$$

$$y(t) = \left(2e^{2(t-t_0)} - 2e^{3(t-t_0)}\right)u(t-t_0)$$

The solution to the system is

$$x(t) = \left(-e^{2(t-t_0)} + 2e^{3(t-t_0)}\right)u(t-t_0)$$

$$y(t) = \left(2e^{2(t-t_0)} - 2e^{3(t-t_0)}\right)u(t-t_0)$$

Convolution

$$ay'' + by' + cy = g(t), \quad y(0) = y_0, \quad y'(0) = y_1$$

Recall the **zero state response** is $\mathscr{L}^{-1}\left\{\frac{G(s)}{as^2+bs+c}\right\}$. We can write this as

$$\mathscr{L}^{-1}\left\{G(s)H(s)\right\},$$

where H is the transfer function[1].

The Zero State Response is the convolution of g and the impulse response h.

If the impulse response is $h(t)$, then the zero state response can be written in terms of a convolution as

$$\mathscr{L}^{-1}\left\{G(s)H(s)\right\} = \int_0^t g(\tau)h(t-\tau)\,d\tau$$

[1] Recall that $H(s)$ is the reciprocol of the characteristic polynomial.

Example

Express the zero state response of the IVP in terms of a convolution.

$$y'' + 100y = g(t), \quad y(0) = y_0, \quad y'(0) = y_1$$

The transfer function $H(s) = \dfrac{1}{\text{Char. poly}} = \dfrac{1}{P(s)}$

$P(s) = s^2 + 100$, $H(s) = \dfrac{1}{s^2+100}$

The impulse response

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+100}\right\} = \frac{1}{10}\sin(10t)$$

The zero state response is

$$\int_0^t g(\tau)\,\frac{1}{10}\sin(10(t-\tau))\,d\tau$$

$$= \frac{1}{10}\int_0^t g(\tau)\sin(10(t-\tau))\,d\tau$$