

6.1 The Determinant

The determinant is a function that assigns a real number to an $n \times n$ matrix. We'll use the notation $\det(A)$ to denote the determinant of the square matrix A .

- ▶ If $A = [a]$ is 1×1 , then $\det(A) = a$.
- ▶ If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is 2×2 , then $\det(A) = ad - bc$.
- ▶ If A is $n \times n$, then A_{ij} is the $(n-1) \times (n-1)$ matrix obtained from A by removing the i^{th} row and j^{th} column.
- ▶ For $n \times n$ matrix A , with $n \geq 2$, $\det(A)$ can be computed by cofactor expansion across the i^{th} row or down the j^{th} column.

$$\det(A) = \sum_{j=1}^n (-1)^{i+j} a_{ij} \det(A_{ij}) = \sum_{i=1}^n (-1)^{i+j} a_{ij} \det(A_{ij}).$$

Triangular Matrices

Triangular matrices have all zero entries below the main diagonal (upper triangular) or above the main diagonal (lower triangular) or both (diagonal).

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

upper triangular

$$\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

lower triangular

A matrix that is both upper triangular and lower triangular is called a **diagonal** matrix.

$$\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

diagonal matrix

Theorem

Let A be an $n \times n$ matrix.

1. If $\vec{0}_n$ is a row vector or a column vector of A , then $\det(A) = 0$.
2. $\det(A^T) = \det(A)$.
3. If A is a triangular matrix (upper, lower or diagonal), the $\det(A)$ is the product of the diagonal entries

$$\det(A) = a_{11}a_{22} \cdots a_{nn}$$

refs are upper triangular. Unfortunately, it is **NOT** true that a matrix A has the same determinant as some $\text{ref}(A)$ or $\text{rref}(A)$. But, we do know how each row operation affects a determinant. So, we can find $\det(A)$ using $\text{rref}(A)$ **if we know all the row operations used to obtain** $\text{rref}(A)$.

Row Operations

Suppose A is an $n \times n$ matrix.

- ▶ If B is obtained from A by performing one row scaling, $kR_i \rightarrow R_i$, then $\det(B) = k \det(A)$.
- ▶ If B is obtained from A by performing one row swap, $R_i \leftrightarrow R_j$, then $\det(B) = -\det(A)$.
- ▶ If B is obtained from A by performing one row replacement, $kR_i + R_j \rightarrow R_j$, then $\det(B) = \det(A)$.

Products

If A and B are $n \times n$ matrices, then

$$\det(AB) = \det(A) \det(B)$$

Examples

Find the products AB and BA where $A = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}$, and

$$B = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}.$$

$$AB = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 10 & 3 \\ 6 & 5 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 13 & -1 \\ 6 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, AB = \begin{bmatrix} 10 & 3 \\ 6 & 5 \end{bmatrix}, BA = \begin{bmatrix} 13 & -1 \\ 6 & 2 \end{bmatrix}$$

Find $\det(A)$, $\det(B)$, $\det(AB)$ and $\det(BA)$.

$$\det(A) = 5(1) - 3(-1) = 8$$

$$\det(B) = 2(2) - 0(1) = 4$$

$$\det(AB) = 10(5) - 6(3) = 50 - 18 = 32$$

$$\det(BA) = 13(2) - 6(-1) = 26 + 6 = 32$$

$$\det(A) \det(B) = 8(4) = 32$$

Invertibility

Let A be an $n \times n$ matrix. Then A is invertible if and only if

$$\det(A) \neq 0.$$

Remark 1: This can be used as a test for invertibility. It's quick for 2×2 or maybe even 3×3 . It's not very practical for larger matrices.

Remark 2: Each row operation changes the determinant by a factor of

- ▶ $k \neq 0$ for scaling,
- ▶ -1 for a row swap,
- ▶ 1 (i.e., no change) for a replacement.

So $\det(A) = (\text{some nonzero number}) \det(\text{rref}(A))$. If A is invertible, then $\text{rref}(A) = I_n$ and $\det(\text{rref}(A)) = 1$. If A is not invertible, then $\text{rref}(A)$ has at least one row of all zero, in which case $\det(\text{rref}(A)) = 0$.

Example

Suppose A is an invertible $n \times n$ matrix. Show that $\det(A^{-1}) = (\det(A))^{-1}$.

$$\det(I_n) = \hat{1} = 1$$

$$A A^{-1} = I_n$$

$$\det(A A^{-1}) = \det(I_n) = 1$$

$$\det(A) \det(A^{-1}) = 1 \quad \text{since } \det(A) \neq 0$$

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

* $\frac{1}{A}$ is nonsense for $n \times n$ matrix A .

Why is $\frac{1}{\det(A)}$ well defined?

$\det(A)$ is a real number.

6.2 Eigenvalues & Eigenvectors

Definition

Let A be an $n \times n$ matrix. An **eigenvalue** of A is a scalar λ for which there exists a nonzero vector $\vec{x}$ such that

$$A\vec{x} = \lambda\vec{x}. \quad (1)$$

For a given eigenvalue λ , a nonzero vector $\vec{x}$ satisfying equation (1) is called an **eigenvector** corresponding to the eigenvalue λ .

Remark 1: Note that eigenvectors are, by definition, **nonzero** vectors.

Remark 2: Eigenvalues are not restricted and can be positive, negative or zero.

Example: $A = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}$

Show that $\vec{x} = \langle 1, 1 \rangle$ is an eigenvector of A by finding an eigenvalue λ such that $A\vec{x} = \lambda\vec{x}$.

$$\begin{aligned} A\vec{x} &= \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \langle 1, 1 \rangle = \langle 5-1, 3+1 \rangle \\ &= \langle 4, 4 \rangle \\ &= 4 \langle 1, 1 \rangle \end{aligned}$$

$$A\vec{x} = 4\vec{x}, \quad \lambda = 4$$

Finding Eigenvalues & Eigenvectors

We saw earlier that if we know that $\lambda = 2$ is an eigenvalue of

$A = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}$, we can find the associated eigenvectors $\vec{x} = t \langle \frac{1}{3}, 1 \rangle$, $t \neq 0$.

We also found that if we know that $\vec{x} = \langle 1, 1 \rangle$ is an eigenvector of A , we can use the same equation, $A\vec{x} = \lambda\vec{x}$, to find the eigenvalue $\lambda = 4$.

Questions:

- ▶ How do we find them without knowing any λ values or vectors up front?
- ▶ Does a matrix A always have eigenvalues and eigenvectors?

No Eigenvalues or Eigenvectors

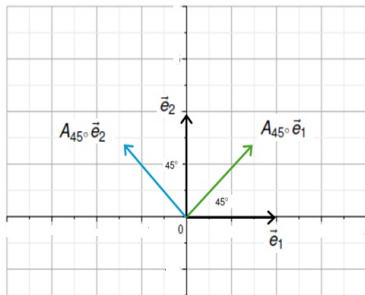


Figure: The matrix $A_{45^\circ} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ rotates each vector 45° counterclockwise.

Question: Could $A_{45^\circ} \vec{x} = \lambda \vec{x}$ for any **nonzero** vector $\vec{x}$ and some number λ ?

$\lambda \vec{x}$ is parallel to $\vec{x}$

$A_{45^\circ} \vec{x}$ can't be parallel to $\vec{x}$, they have a 45° angle between them

The Characteristic Equation

How do we actually find these numbers and vectors? We actually start by finding the eigenvalues. Let's derive a way to find λ such that

$$A\vec{x} = \lambda\vec{x}$$

$$\text{Use } I_n \vec{x} = \vec{x} \quad A\vec{x} = \lambda\vec{x} \Rightarrow A\vec{x} = \lambda I_n \vec{x}$$

$$\Rightarrow A\vec{x} - \lambda I_n \vec{x} = \vec{0}_n$$

$(A - \lambda I_n)\vec{x} = \vec{0}_n$ we need non trivial solutions, so $A - \lambda I_n$ must not be invertible. This requires

$$\det(A - \lambda I_n) = 0.$$

A scalar valued equation for λ .

Definition

Let A be an $n \times n$ matrix. The function

$$P_A(\lambda) = \det(A - \lambda I_n)$$

is called the **characteristic polynomial** of the matrix A . The equation

$$P_A(\lambda) = 0, \quad \text{i.e.,} \quad \det(A - \lambda I_n) = 0$$

is called the **characteristic equation** of the matrix A .

A the name suggests, $P_A(\lambda)$ is always a polynomial in λ . The degree matches the size of the matrix, n , and the leading coefficient is 1 if n is even and -1 if n is odd.

Example

Find the characteristic polynomial of $A = \begin{bmatrix} 4 & 3 & -1 \\ 1 & 2 & 2 \\ 0 & 0 & -3 \end{bmatrix}$.

$$\det(A - \lambda I_3)$$

$$A - \lambda I_3 = \begin{bmatrix} 4 & 3 & -1 \\ 1 & 2 & 2 \\ 0 & 0 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 3 & -1 \\ 1 & 2 & 2 \\ 0 & 0 & -3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 4-\lambda & 3 & -1 \\ 1 & 2-\lambda & 2 \\ 0 & 0 & -3-\lambda \end{bmatrix}.$$

$$\det(A - \lambda I_3) = +0 \det \begin{bmatrix} 3 & -1 \\ 2-\lambda & 2 \end{bmatrix} - 0 \det \begin{bmatrix} 4-\lambda & -1 \\ 1 & 2 \end{bmatrix} \\ + (-3-\lambda) \det \begin{bmatrix} 4-\lambda & 3 \\ 1 & 2-\lambda \end{bmatrix}$$

$$= (-3-\lambda) \left[(4-\lambda)(2-\lambda) - 1(3) \right]$$

$$= -(3+\lambda) \left[8 - 6\lambda + \lambda^2 - 3 \right]$$

$$= -(3+\lambda) (\lambda^2 - 6\lambda + 5)$$

$$= -(3+\lambda) (\lambda - 5) (\lambda - 1)$$

Theorem

Let A be an $n \times n$ matrix, and let $P_A(\lambda)$ be the characteristic polynomial of A . The number λ_0 is an eigenvalue of A if and only if $P_A(\lambda_0) = 0$. That is, λ_0 is an eigenvalue of A if and only if it is a root of the characteristic equation $\det(A - \lambda I_n) = 0$.

Finding the eigenvalues can be challenging if A is a large matrix (high degree polynomial). Once an eigenvalue is known, we can find eigenvectors by solving the homogeneous equation

$$(A - \lambda I_n)\vec{x} = \vec{0}_n$$

using row reduction. This means that the eigenvectors are the nonzero vectors in $\mathcal{N}(A - \lambda I_n)$.

Example $A = \begin{bmatrix} 4 & 3 & -1 \\ 1 & 2 & 2 \\ 0 & 0 & -3 \end{bmatrix}$

The characteristic polynomial was

$$P_A(\lambda) = -(3 + \lambda)(\lambda - 5)(\lambda - 1) = -\lambda^3 + 3\lambda^2 + 13\lambda - 15.$$

Find an eigenvector for each eigenvalue.

$$P_A(\lambda) = 0 \Rightarrow -(3 + \lambda)(\lambda - 5)(\lambda - 1) = 0$$
$$\Rightarrow \lambda_1 = -3, \lambda_2 = 5, \lambda_3 = 1$$

$$\text{For } \lambda_2 = 5, \quad A - 5I_3 = \begin{bmatrix} 4-5 & 3 & -1 \\ 1 & 2-5 & 2 \\ 0 & 0 & -3-5 \end{bmatrix}$$
$$= \begin{bmatrix} -1 & 3 & -1 \\ 1 & -3 & 2 \\ 0 & 0 & -8 \end{bmatrix}$$

Now, we solve $(A - 5I_3)\vec{x} = \vec{0}_3$

$$A = \begin{bmatrix} 4 & 3 & -1 \\ 1 & 2 & 2 \\ 0 & 0 & -3 \end{bmatrix}$$

$$\text{ref}(A - sI_3) = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

for solution $\vec{x} = \langle x_1, x_2, x_3 \rangle$

$$\begin{aligned} x_1 &= 3x_2, \\ x_2 &\text{ is free} \\ x_3 &= 0 \end{aligned}$$

Eigenvectors are $\vec{x} = x_2 \langle 3, 1, 0 \rangle$

an example is $\vec{x}_2 = \langle 3, 1, 0 \rangle$