November 21 Math 3260 sec. 53 Fall 2025

6.1 The Determinant

The determinant is a function that assigns a real number to an $n \times n$ matrix. We'll use the notation $\det(A)$ to denote the determinant of the square matrix A.

- If $A = [a]$ is 1×1, then $\det(A) = a$.
- If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is 2×2, then $\det(A) = ad - bc$.
- If A is $n \times n$, then A_{ij} is the $(n-1) \times (n-1)$ matrix obtained from A by removing the i^{th} row and j^{th} column.
- For $n \times n$ matrix A, with $n \geq 2$, $\det(A)$ can be computed by cofactor expansion across the i^{th} row or down the j^{th} column.

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}).$$

Triangular Matrices

Triangular matrices have all zero entries below the main diagonal (upper triangular) or above the main diagonal (lower triangular) or both (diagonal).

$$\underset{\text{upper triangular}}{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}} \quad \underset{\text{lower triangular}}{\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}}$$

A matrix that is both upper triangular and lower triangular is called a **diagonal** matrix.

$$\underset{\text{diagonal matrix}}{\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}}$$

Theorem

Let A be an $n \times n$ matrix.

1. If $\vec{0}_n$ is a row vector or a column vector of A, then $\det(A) = 0$.

2. $\det(A^T) = \det(A)$.

3. If A is a triangular matrix (upper, lower or diagonal), the $\det(A)$ is the product of the diagonal entries

$$\det(A) = a_{11}a_{22}\cdots a_{nn}$$

refs are upper triangular. Unfortunately, it is **NOT** true that a matrix A has the same determinant as some $\mathrm{ref}(A)$ or $\mathrm{rref}(A)$. But, we do know how each row operation affects a determinant. So, we can find $\det(A)$ using $\mathrm{rref}(A)$ **if we know all the row operations used to obtain $\mathrm{rref}(A)$.**

Row Operations

Suppose A is an $n \times n$ matrix.

- If B is obtained from A by performing one row scaling, $kR_i \to R_i$, then $\det(B) = k\det(A)$.
- If B is obtained from A by performing one row swap, $R_i \leftrightarrow R_j$, then $\det(B) = -\det(A)$.
- If B is obtained from A by performing one row replacement, $kR_i + R_j \to R_j$, then $\det(B) = \det(A)$.

Products

If A and B are $n \times n$ matrices, then

$$\det(AB) = \det(A)\det(B)$$

Examples

Find the products AB and BA where $A = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}$, and $B = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$.

$$AB = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 10 & 3 \\ 6 & 5 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 13 & -1 \\ 6 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \; AB = \begin{bmatrix} 10 & 3 \\ 6 & 5 \end{bmatrix}, \; BA = \begin{bmatrix} 13 & -1 \\ 6 & 2 \end{bmatrix}$$

Find $\det(A)$, $\det(B)$, $\det(AB)$ and $\det(BA)$.

$\det(A) = 5(1) - 3(-1) = 8$

$\det(B) = 2(2) - 0(1) = 4$

$\det(AB) = 10(5) - 6(3) = 50 - 18 = 32$

$\det(BA) = 13(2) - 6(-1) = 26 + 6 = 32$

$\det(A)\det(B) = 8(4) = 32$

Invertibility

Let A be an $n \times n$ matrix. Then A is invertible if and only if

$$\det(A) \neq 0.$$

Remark 1: This can be used as a test for invertibility. It's quick for 2×2 or maybe even 3×3. It's not very practical for larger matrices.

Remark 2: Each row operation changes the determinant by a factor of

- $k \neq 0$ for scaling,
- -1 for a row swap,
- 1 (i.e., no change) for a replacement.

So $\det(A) = $ (some nonzero number) $\det(\text{rref}(A))$. If A is invertible, then $\text{rref}(A) = I_n$ and $\det(\text{rref}(A)) = 1$. If A is not invertible, then $\text{rref}(A)$ has at least one row of all zero, in which case $\det(\text{rref}(A)) = 0$.

Example

Suppose A is an invertible $n \times n$ matrix. Show that $\det(A^{-1}) = (\det(A))^{-1}$.

$\det(I_n) = 1^n = 1.$

$A A^{-1} = I_n$

$\det(A A^{-1}) = \det(I_n) = 1$

$\det(A) \det(A^{-1}) = 1 \qquad \det(A) \neq 0$

$\det(A^{-1}) = \dfrac{1}{\det(A)}.$

* For $n \times n$ matrix A, we said

$\frac{1}{A}$ is meaningless

why is $\frac{1}{\det(A)}$ well defined?

$\det(A)$ is a scalar, i.e., a real number.

6.2 Eigenvalues & Eigenvectors

Definition

Let A be an $n \times n$ matrix. An **eigenvalue** of A is a scalar λ for which there exists a nonzero vector $\vec{x}$ such that

$$A\vec{x} = \lambda\vec{x}. \tag{1}$$

For a given eigenvalue λ, a nonzero vector $\vec{x}$ satisfying equation (1) is called an **eigenvector** corresponding to the eigenvalue λ.

Remark 1: Note that eigenvectors are, by definition, **nonzero** vectors.

Remark 2: Eigenvalues are not restricted and can be positive, negative or zero.

Example: $A = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}$

Show that $\vec{x} = \langle 1, 1 \rangle$ is an eigenvector of A by finding an eigenvalue λ such that $A\vec{x} = \lambda\vec{x}$.

$$A\vec{x} = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \langle 1, 1 \rangle = \langle 5-1, 3+1 \rangle$$

$$= \langle 4, 4 \rangle$$

$$= 4 \langle 1, 1 \rangle$$

So $\langle 1, 1 \rangle$ is an eigenvector for eigenvalue $\lambda = 4$.

Finding Eigenvalues & Eigenvectors

We saw earlier that if we know that $\lambda = 2$ is an eigenvalue of $A = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix}$, we can find the associated eigenvectors $\vec{x} = t\left\langle \frac{1}{3}, 1 \right\rangle$, $t \neq 0$.

We also found that if we know that $\vec{x} = \langle 1, 1 \rangle$ is an eigenvector of A, we can use the same equation, $A\vec{x} = \lambda\vec{x}$, to find the eigenvalue $\lambda = 4$.

Questions:

- How do we find them without knowing any λ values or vectors up front?
- Does a matrix A always have eigenvalues and eigenvectors?

No Eigenvalues or Eigenvectors

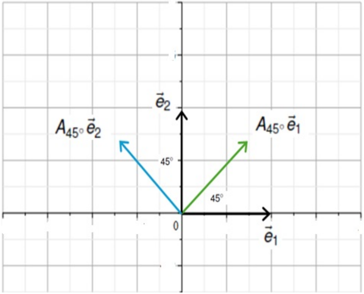

Figure: The matrix $A_{45^\circ} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ rotates each vector 45° counterclockwise.

Question: Could $A_{45^\circ}\vec{x} = \lambda\vec{x}$ for any **nonzero** vector $\vec{x}$ and some number λ?

$\lambda\vec{x}$ is parallel to $\vec{x}$

$A_{45^\circ}\vec{x}$ and $\vec{x}$ make a 45° angle, they can't be parallel

The Characteristic Equation

How do we actually find these numbers and vectors? We actually start by finding the eigenvalues. Let's derive a way to find λ such that

$$A\vec{x} = \lambda\vec{x}$$

use $\vec{x} = I_n\vec{x}$, $A\vec{x} = \lambda\vec{x} \Rightarrow A\vec{x} = \lambda I_n\vec{x}$

$\Rightarrow A\vec{x} - \lambda I_n\vec{x} = \vec{0}_n \Rightarrow (A - \lambda I_n)\vec{x} = \vec{0}_n$

we need nontrivial solutions. This requires the matrix $A - \lambda I_n$ be noninvertible.

This requires $\det(A - \lambda I_n) = 0$.

This is a scalar equation for the scalar λ.

Definition

Let A be an $n \times n$ matrix. The function

$$P_A(\lambda) = \det(A - \lambda I_n)$$

is called the **characteristic polynomial** of the matrix A. The equation

$$P_A(\lambda) = 0, \quad \text{i.e.,} \quad \det(A - \lambda I_n) = 0$$

is called the **characteristic equation** of the matrix A.

A the name suggests, $P_A(\lambda)$ is always a polynomial in λ. The degree matches the size of the matrix, n, and the leading coefficient is 1 if n is even and -1 if n is odd.

Example

Find the characteristic polynomial of $A = \begin{bmatrix} 4 & 3 & -1 \\ 1 & 2 & 2 \\ 0 & 0 & -3 \end{bmatrix}$.

$\det(A - \lambda I_3)$

$$A - \lambda I_3 = \begin{bmatrix} 4 & 3 & -1 \\ 1 & 2 & 2 \\ 0 & 0 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 3 & -1 \\ 1 & 2 & 2 \\ 0 & 0 & -3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 4-\lambda & 3 & -1 \\ 1 & 2-\lambda & 2 \\ 0 & 0 & -3-\lambda \end{bmatrix}$$

Using the third row

$$\det(A-\lambda I_3) = +0\det\begin{bmatrix}3 & -1\\ 2-\lambda & 2\end{bmatrix} - 0\det\begin{bmatrix}4-\lambda & -1\\ 1 & 2\end{bmatrix} + (-3-\lambda)\det\begin{bmatrix}4-\lambda & 3\\ 1 & 2-\lambda\end{bmatrix}$$

$$= (-3-\lambda)\left[(4-\lambda)(2-\lambda) - 1(3)\right]$$

$$= -(3+\lambda)\left[8 - 6\lambda + \lambda^2 - 3\right]$$

$$= -(3+\lambda)(\lambda^2 - 6\lambda + 5)$$

$$= -(3+\lambda)(\lambda-5)(\lambda-1)$$

$$P_A(\lambda) = -(3+\lambda)(\lambda-5)(\lambda-1)$$

Theorem

Let A be an $n \times n$ matrix, and let $P_A(\lambda)$ be the characteristic polynomial of A. The number λ_0 is an eigenvalue of A if and only if $P_A(\lambda_0) = 0$. That is, λ_0 is an eigenvalue of A if and only if it is a root of the characteristic equation $\det(A - \lambda I_n) = 0$.

Finding the eigenvalues can be challenging if A is a large matrix (high degree polynomial). Once an eigenvalue is known, we can find eigenvectors by solving the homogeneous equation

$$(A - \lambda I_n)\vec{x} = \vec{0}_n$$

using row reduction. This means that the eigenvectors are the nonzero

vectors in $\mathcal{N}(A - \lambda I_n)$.

Example $A = \begin{bmatrix} 4 & 3 & -1 \\ 1 & 2 & 2 \\ 0 & 0 & -3 \end{bmatrix}$

The characteristic polynomial was

$$P_A(\lambda) = -(3+\lambda)(\lambda-5)(\lambda-1) = -\lambda^3 + 3\lambda^2 + 13\lambda - 15.$$

Find an eigenvector for each eigenvalue.

$$P_A(\lambda) = 0 = -(3+\lambda)(\lambda-5)(\lambda-1)$$

$$\Rightarrow \quad \lambda_1 = -3, \quad \lambda_2 = 5, \quad \lambda_3 = 1.$$

λ_2

$$A - 5I_3 = \begin{bmatrix} 4-5 & 3 & -1 \\ 1 & 2-5 & 2 \\ 0 & 0 & -3-5 \end{bmatrix} = \begin{bmatrix} -1 & 3 & -1 \\ 1 & -3 & 2 \\ 0 & 0 & -8 \end{bmatrix}$$

$$\text{rref}(A - 5I_3) = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 4 & 3 & -1 \\ 1 & 2 & 2 \\ 0 & 0 & -3 \end{bmatrix}$$

If a solution $\vec{x} = \langle x_1, x_2, x_3 \rangle$

then $x_1 = 3x_2$
x_2 is free
$x_3 = 0$

So eigenvectors look like

$\vec{x} = x_2 \langle 3, 1, 0 \rangle$ for $x_2 \neq 0$

an example of an eigen vector associated w/ eigen value $\lambda_2 = 5$ is $\vec{x}_2 = \langle 3, 1, 0 \rangle$