November 27 Math 2306 sec. 52 Spring 2023

Section 16: Laplace Transforms of Derivatives and IVPs

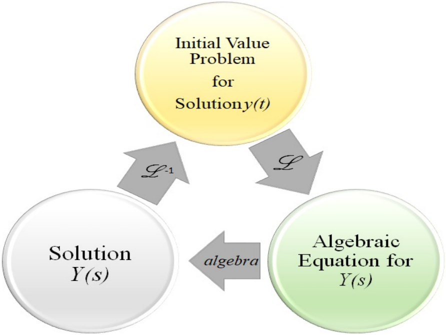


Figure: We'll use the Laplace transform as a tool for solving certain IVPs and systems of IVPs. Our use will be restricted to IVPs with **constant coefficients** and initial conditions given at $t = 0$.

The Laplace Transform of Derivatives

For $y = y(t)$ defined on $[0, \infty)$ having derivatives y', y'' and so forth, if $\mathscr{L}\{y(t)\} = Y(s)$, then

$$
\begin{aligned}
\mathscr{L}\left\{\frac{dy}{dt}\right\} &= sY(s) - y(0) \\
\mathscr{L}\left\{\frac{d^2y}{dt^2}\right\} &= s^2Y(s) - sy(0) - y'(0) \\
\mathscr{L}\left\{\frac{d^3y}{dt^3}\right\} &= s^3Y(s) - s^2y(0) - sy'(0) - y''(0) \\
&\vdots \\
\mathscr{L}\left\{\frac{d^ny}{dt^n}\right\} &= s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - y^{(n-1)}(0).
\end{aligned}
$$

Use Laplace Transforms to Solve and IVP

- Start with constant coefficient IVP with IC at $t = 0$. For example[a]

$$ay'' + by' + cy = g(t), \quad y(0) = y_0, \quad y'(0) = y_1.$$

- Let $Y(s) = \mathscr{L}\{y(t)\}$ and take the transform of both sides of the ODE using any necessary results.
- Sub in the initial conditions where they appear in the transformed equation.
- Use basic algebra to isolate the transform $Y(s)$.
- Using whatever algebra or function identities that are needed, take the inverse transform to obtain the solution

$$y(t) = \mathscr{L}^{-1}\{Y(s)\}.$$

[a] The IVP can be of any order.

The Unit Impulse

The last example we looked at was a circuit problem with a piecewise constant voltage applied over a specific time interval $[t_0, t_1]$.

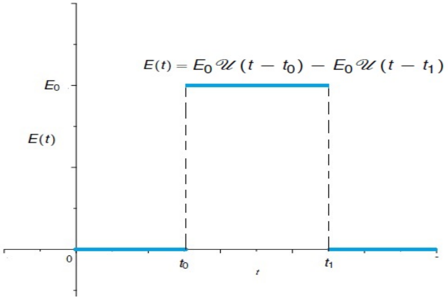


Figure: The current in the circuit satisfied $L\frac{di}{dt} + Ri = E_0\mathscr{U}(t-t_0) - E_0\mathscr{U}(t-t_1)$

The Unit Impulse

In engineering applications, it is useful to have a model of a force (or signal) that is applied over an infinitesimal time interval. That is, we would like to model this process in the limit $t_1 \to t_0$ while keeping the total *magnitude* or *strength* (its integral) of the force fixed.

We can build such a model by considering rectangular[1] functions and reducing the width while keeping the area fixed.

[1] The shape doesn't have to be a rectangle. It could be triangles, or a hump of a cosine, or something else.

The Unit Impulse

In order to build up to the definition of our unit impulse, we introduce the family of piecewise constant, rectangular functions $R_\epsilon(t) = \begin{cases} \frac{1}{2\epsilon}, & |t| < \epsilon \\ 0, & |t| > \epsilon \end{cases}$.

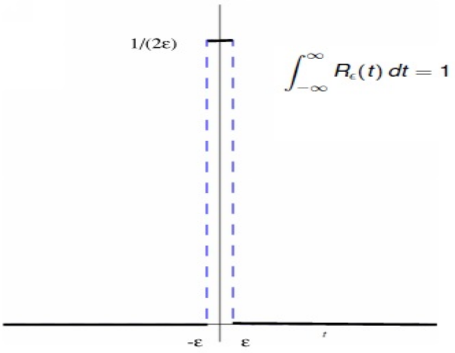


Figure: The value of ϵ determines the height and width of the rectangle. But for every $\epsilon > 0$, the integral of R_ϵ over the real line is 1.

Unit Impulse

We can plot R_ϵ for various values of ϵ and see that as ϵ gets smaller, the rectangle gets narrow and tall. But the area of the rectangle is kept constant at 1.

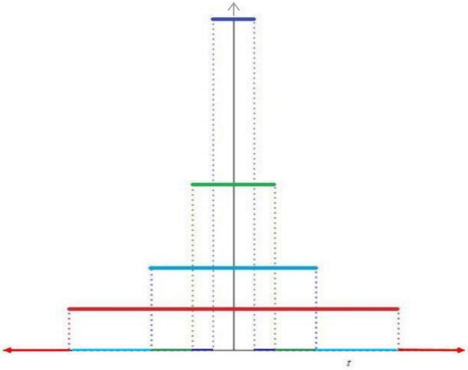


Figure: $R_\epsilon(t) = \begin{cases} \frac{1}{2\epsilon}, & |t| < \epsilon \\ 0, & |t| > \epsilon \end{cases}$

Unit Impulse

The Dirac delta *function*, denoted by $\delta(\cdot)$, models a strong instantaneous force. One way to define this function is as the limit

$$\delta(t) = \lim_{\epsilon \to 0} R_\epsilon(t).$$

More generally, we can move the location of the force to occur at time $t = a$ for some $a > 0$. We express this as $\delta(t-a)$ and refer to it as a **unit impulse at a** or (centered at a).

Unit Impulse $\delta(t - a)$

The Dirac delta function is a limit of traditional functions, but it isn't really a function (in the input-output sense). It is an example of what is called a *generalized function*, a *functional*, or a *distribution*. It is a mathematical object whose properties are defined in combination with integration. We can think of it as acting on continuous functions in specific ways.

The following hold:

- $\displaystyle\int_{-\infty}^{\infty} \delta(t - a)\, dt = 1$ for any real number a.
- $\displaystyle\int_{-\infty}^{\infty} \delta(t - a) f(t)\, dt = f(a)$ if f is continuous at a.
- $\mathscr{L}\{\delta(t - a)\} = e^{-as}$ for any constant $a \geq 0$.
- In the sense of distributions, it is related to $\mathscr{U}$ via $\dfrac{d}{dt}\mathscr{U}(t - a) = \delta(t - a)$.

Delta as a Model of a Unit Impulse

The *function* $\delta(t)$ is used as a model of a force of magnitude 1 applied instantaneously at time $t = 0$. Hence a function $f(t) = f_0\delta(t - a)$ can be used to model a force of magnitude f_0 applied instantaneously at the time $t = a$.

For example, suppose our LR series circuit has zero applied voltage for $t \neq t_0$. A switch is closed and opened immediately to apply a voltage E_0 at $t = t_0$. The differential equation modeling the charge on the capacitor is given by

$$L\frac{di}{dt} + Ri = E_0\delta(t - t_0).$$

Remark

We can't work with the Dirac delta function the way we might work with other forcing functions (e.g., exponentials or sines and cosines). But we do know what the Laplace transform of $\delta(t - t_0)$ is, so we will be able to solve IVPs that involve differential equations of the form shown here.

Solve the IVP using the Laplace Transform

A 1 kg mass is suspended from a spring with spring constant 10 N/m. A damper induces damping of 6 N per m/sec of velocity. The object starts from rest at equilibrium. At time $t = 1$ second, a unit impulse force is applied to the object. Determine the object's position for $t > 0$.

The corresponding IVP for the situation described is

$$x'' + 6x' + 10x = \delta(t-1), \quad x(0) = 0, \quad x'(0) = 0$$

Let $X(s) = \mathcal{L}\{x(t)\}$. Take $\mathcal{L}$ of the ODE

$$\mathcal{L}\{x'' + 6x' + 10x\} = \mathcal{L}\{\delta(t-1)\}$$

$$\mathcal{L}\{x''\} + 6\mathcal{L}\{x'\} + 10\mathcal{L}\{x\} = e^{-1s}$$

Remember the model is $mx'' + bx' + kx = f(t)$ with initial position $x(0)$ and initial velocity $x'(0)$.

$$s^2X(s) - s\underset{0}{\underbrace{x(0)}} - \underset{0}{\underbrace{x'(0)}} + 6\left(sX(s) - \underset{0}{\underbrace{x(0)}}\right) + 10X(s) = e^{-s}$$

$$(s^2+6s+10)X(s) = e^{-s}$$

Characteristic polynomial

$$X(s) = \frac{e^{-s}}{s^2+6s+10} = e^{-s}\,\frac{1}{s^2+6s+10}$$

We need to find

$$f(t) = \mathscr{L}^{-1}\left\{\frac{1}{s^2+6s+10}\right\}.$$

Since $s^2+6s+10$ doesn't factor, we'll complete the square.

$$s^2+6s+10 = s^2+6s+9-9+10 = (s+3)^2+1$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2+1}\right\} = e^{-3t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$s-a = s+3$ $\quad \frac{1}{s^2+1}$ $\quad = e^{-3t}\sin t$

$a = -3$

our solution will be $f(t-1)\mathcal{U}(t-1)$

The position $x(t) = \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{e^{-1s}\frac{1}{(s+3)^2+1}\right\}$

$$\boxed{x(t) = e^{-3(t-1)}\sin(t-1)\,\mathcal{U}(t-1)}$$

$$x'' + 6x' + 10x = \delta(t-1), \quad x(0) = 0, \quad x'(0) = 0$$

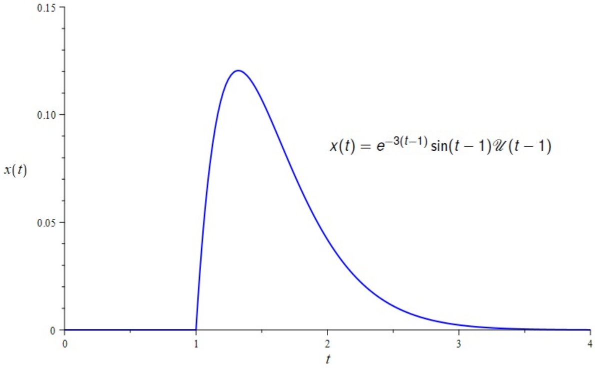


Figure: Graph of the solution to the IVP with unit impulse external force at $t = 1$.

Transfer Function & Impulse Response

$$ay'' + by' + cy = g(t), \tag{1}$$

Definition

The function $H(s) = \dfrac{1}{as^2 + bs + c}$ is called the **transfer function** for the differential equation (2).

Remark 1: The transfer function is the Laplace transform of the solution to the IVP

$$ay'' + by' + cy = \delta(t), \quad y(0) = 0, \quad y'(0) = 0.$$

Transfer Function & Impulse Response

$$ay'' + by' + cy = g(t), \tag{2}$$

Definition

The **impulse response** function, $h(t)$, for the differential equation (2) is the inverse Laplace transform of the transfer function

$$h(t) = \mathscr{L}^{-1}\{H(s)\} = \mathscr{L}^{-1}\left\{\frac{1}{as^2 + bs + c}\right\}.$$

Remark 2: The impulse response is the solution to the IVP

$$ay'' + by' + cy = \delta(t), \quad y(0) = 0, \quad y'(0) = 0.$$

Convolution

$$ay'' + by' + cy = g(t), \quad y(0) = y_0, \quad y'(0) = y_1$$

Recall the **zero state response** is $\mathscr{L}^{-1}\left\{\dfrac{G(s)}{as^2+bs+c}\right\}$. We can write this as

$$\mathscr{L}^{-1}\left\{G(s)H(s)\right\},$$

where H is the transfer function.

The Zero State Response is the convolution of g and the impulse response h.

If the impulse response is $h(t)$, then the zero state response can be written in terms of a convolution as

$$\mathscr{L}^{-1}\left\{G(s)H(s)\right\} = \int_0^t g(\tau)h(t-\tau)\,d\tau$$

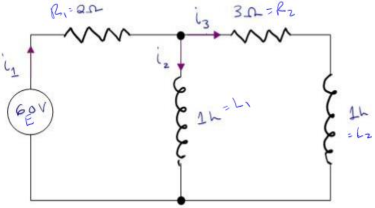
Solving a System

We can solve a system of ODEs using Laplace transforms. Here, we'll consider systems that are

- linear,
- having initial conditions at $t = 0$, and
- constant coefficient.

Let's see it in action (i.e. with a couple of examples).

Example



$$
\begin{aligned}
L_1 i_2' &+ R_1(i_2+i_3) & &= E \\
L_2 i_3' &+ R_1(i_2+i_3) &+ R_2 i_3 &= E
\end{aligned}
$$

Figure: If we label current i_2 as $x(t)$ and current i_3 as $y(t)$, we get the system of equations below. (Assuming $i_1(0) = 0$.)

Solve the system of equations

$$
\begin{aligned}
\frac{dx}{dt} &= -2x - 2y + 60, \quad x(0) = 0 \\
\frac{dy}{dt} &= -2x - 5y + 60, \quad y(0) = 0
\end{aligned}
$$

Let $X(s) = \mathcal{L}\{x(t)\}$ and $Y(s) = \mathcal{L}\{y(t)\}$.

$$\frac{dx}{dt} = -2x - 2y + 60, \quad x(0) = 0$$
$$\frac{dy}{dt} = -2x - 5y + 60, \quad y(0) = 0$$

Take $\mathcal{L}$ of both ODEs

$$\mathcal{L}\{x'\} = \mathcal{L}\{-2x - 2y + 60\}$$

$$\mathcal{L}\{x'\} = -2\mathcal{L}\{x\} - 2\mathcal{L}\{y\} + 60\mathcal{L}\{1\}$$

$$sX(s) - \underbrace{x(0)}_{0} = -2X(s) - 2Y(s) + \frac{60}{s}$$

$$\mathcal{L}\{y'\} = \mathcal{L}\{-2x - 5y + 60\}$$

$$sY(s) - \underbrace{y(0)}_{0} = -2X(s) - 5Y(s) + \frac{60}{s}$$

we want to write this system like

$$a X(s) + b Y(s) = e$$

$$c X(s) + d Y(s) = f$$

$$s X(s) + 2 X(s) + 2 Y(s) = \frac{60}{s}$$

$$s Y(s) + 2 X(s) + 5 Y(s) = \frac{60}{s}$$

$$(s+2) X(s) + 2 Y(s) = \frac{60}{s}$$

$$2 X(s) + (s+5) Y(s) = \frac{60}{s}$$

written in matrix format

$$\begin{bmatrix} s+2 & 2 \\ 2 & s+5 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 60/s \\ 60/s \end{bmatrix}$$

Let $A = \begin{bmatrix} s+2 & 2 \\ 2 & s+5 \end{bmatrix}$, $A_x = \begin{bmatrix} 60/s & 2 \\ 60/s & s+5 \end{bmatrix}$, $A_y = \begin{bmatrix} s+2 & 60/s \\ 2 & 60/s \end{bmatrix}$

$$\det(A) = (s+2)(s+5) - 2\cdot 2 = s^2 + 7s + 10 - 4 = s^2 + 7s + 6 = (s+1)(s+6)$$

$$\det(A_x) = \frac{60}{s}(s+5) - \frac{60}{s}(2) = \frac{60}{s}(s+5-2) = \frac{60}{s}(s+3)$$

$$\det(A_y) = \frac{60}{s}(s+2) - \frac{60}{s}(2) = \frac{60}{s}(s+2-2) = \frac{60}{s}(s) = 60$$

$$X(s) = \frac{\det(A_x)}{\det(A)} = \frac{\frac{60}{s}(s+3)}{(s+1)(s+6)} = \frac{60\ (s+3)}{s(s+1)(s+6)}$$

$$Y(s) = \frac{\det(A_y)}{\det(A)} = \frac{60}{(s+1)(s+6)}$$

These require partial fractions. Doing a PFD

$$X(s) = \frac{30}{s} - \frac{24}{s+1} - \frac{6}{s+6}$$

$$Y(s) = \frac{12}{s+1} - \frac{12}{s+6}$$

The solution $x(t) = \mathcal{L}^{-1}\{X(s)\}$ and $y(t) = \mathcal{L}^{-1}\{Y(s)\}$.

$$\boxed{\begin{aligned} x(t) &= 30 - 24e^{-t} - 6e^{-6t} \\ y(t) &= 12e^{-t} - 12e^{-6t} \end{aligned}}$$