November 28 Math 2306 sec. 51 Fall 2022

Section 17: Fourier Series: Trigonometric Series

Suppose $f(x)$ is defined for $-\pi < x < \pi$. We would like to know how to write f as a series **in terms of sines and cosines.**

Task: Find coefficients (numbers) a_0, $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ such that[1]

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right).$$

[1] We'll write $\frac{a_0}{2}$ as opposed to a_0 purely for convenience.

Finding an Example Coefficient

Let's find the coefficient b_4.

Start with the series $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$, and multiply both sides by $\sin(4x)$.

$$f(x)\sin(4x) = \frac{a_0}{2}\sin(4x) + \sum_{n=1}^{\infty}(a_n \cos nx \sin(4x) + b_n \sin nx \sin(4x)).$$

Using the orthogonality property of these functions, we integrated from $-\pi$ to π. Only one term on the right side was nonzero giving us a formula for b_4

$$b_4 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(4x)\,dx$$

Finding Fourier Coefficients

Note that there was nothing special about seeking the 4^{th} sine coefficient b_4. We could have just as easily sought b_m for any positive integer m. We would simply start by introducing the factor $\sin(mx)$.

Moreover, using the same orthogonality property, we could pick on the a's by starting with the factor $\cos(mx)$—including the constant term since $\cos(0 \cdot x) = 1$. The only minor difference we want to be aware of is that

$$\int_{-\pi}^{\pi} \cos^2(mx)\,dx = \begin{cases} 2\pi, & m = 0 \\ \pi, & m \geq 1 \end{cases}$$

Careful consideration of this sheds light on why it is conventional to take the constant to be $\frac{a_0}{2}$ as opposed to just a_0.

The Fourier Series of $f(x)$ on $(-\pi, \pi)$

The **Fourier series** of the function f defined on $(-\pi, \pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \, .$$

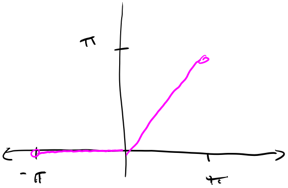
Where

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx, \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx, \quad \text{and} \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \end{aligned}$$

Example

Find the Fourier series of the piecewise defined function

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 \le x < \pi \end{cases}$$



Find a_0

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left(\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi} x\,dx\right)$$

$$= \frac{1}{\pi}\left(\frac{x^2}{2}\right)\Bigg|_0^{\pi} = \frac{1}{\pi}\left(\frac{\pi^2}{2} - \frac{0^2}{2}\right) = \frac{\pi}{2}$$

$$\boxed{a_0 = \frac{\pi}{2}}$$

Find a_n:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$= \frac{1}{\pi}\left(\int_{-\pi}^{0} 0 \cdot \cos(nx)\,dx + \int_{0}^{\pi} x\cos(nx)\,dx \right)$$

$$= \frac{1}{\pi}\int_{0}^{\pi} x\cos(nx)\,dx$$

Int by parts

$u = x \quad du = dx$

$dv = \cos(nx)\,dx$

$v = \frac{1}{n}\sin(nx)$

$$= \frac{1}{\pi}\left[\frac{x\sin(nx)}{n}\Big|_0^{\pi} - \int_0^{\pi} \frac{1}{n}\sin(nx)\,dx \right]$$

$$= \frac{1}{\pi}\left[\pi\sin(n\pi) - 0\sin(0) + \frac{1}{n^2}\cos(nx) \Big|_0^{\pi} \right.$$

$\sin(n\pi) = 0$ for all $n = 1, 2, 3, \ldots$

$$= \frac{1}{\pi}\left[\frac{1}{n^2}\cos(n\pi) - \frac{1}{n^2}\cos(0)\right]$$

$$\cos(n\pi) = \begin{cases} 1, & n \text{ is even} \\ -1, & n \text{ is odd} \end{cases} = (-1)^n$$

$$\boxed{a_n = \frac{(-1)^n - 1}{n^2\pi}}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

$$= \frac{1}{\pi}\left(\int_{-\pi}^{0} 0\cdot\sin(nx)\,dx + \int_{0}^{\pi} x\sin(nx)\,dx\right)$$

$$= \frac{1}{\pi}\int_0^{\pi} x\,\sin(nx)\,dx$$

By parts

$u = x \qquad du = dx$

$dv = \sin(nx)\,dx$

$v = \frac{-1}{n}\cos(nx)$

$$= \frac{1}{\pi}\left[\frac{-1}{n}x\cos(nx)\Big|_0^{\pi} - \int_0^{\pi}\frac{-1}{n}\cos(nx)\,dx\right.$$

$$= \frac{1}{\pi}\left[\frac{-1}{n}\pi\cos(n\pi) - \frac{-1}{n}\cdot 0\cos(0) + \frac{1}{n^2}\sin(nx)\Big|_0^{\pi}\right.$$

$$= \frac{1}{\pi}\left(\frac{-\pi}{n}\cos(n\pi) + \frac{1}{n^2}\sin(n\pi) - \frac{1}{n^2}\sin(0)\right)$$

$$= \frac{1}{\pi}\left(\frac{-\pi}{n}\right)(-1)^n = \frac{-1}{n}(-1)^n$$

$$= \frac{1}{n}(-1)(-1)^n = \frac{1}{n}(-1)^{n+1}$$

$$\boxed{b_n = \frac{(-1)^{n+1}}{n}} \qquad \boxed{a_n = \frac{(-1)^n - 1}{n^2 \pi}}$$

$$\boxed{a_0 = \frac{\pi}{2}} \qquad \frac{a_0}{2} = \frac{\frac{\pi}{2}}{2} = \frac{\pi}{4}$$

$$\boxed{f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2 \pi} \cos(nx) + \frac{(-1)^{n+1}}{n} \sin(nx)}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right).$$

An Orthogonal Set of Functions on $[-p, p]$

This set can be generalized by using a simple change of variables $t = \frac{\pi x}{p}$ to obtain the orthogonal set on $[-p, p]$

$$\left\{1, \cos\frac{n\pi x}{p}, \sin\frac{m\pi x}{p} \,\middle|\, n, m = \pm 1, \pm 2, \dots\right\}$$

There are many interesting and useful orthogonal sets of functions (on appropriate intervals). What follows is readily extended to other such (infinite) sets.

Fourier Series on an interval $(-p, p)$

The set of functions $\left\{1, \cos\left(\frac{n\pi x}{p}\right), \sin\left(\frac{m\pi x}{p}\right) | n, m \geq 1\right\}$ is orthogonal on $[-p, p]$. Moreover, we have the properties

$$\int_{-p}^{p} \cos\left(\frac{n\pi x}{p}\right)\, dx = 0 \quad \text{and} \quad \int_{-p}^{p} \sin\left(\frac{m\pi x}{p}\right)\, dx = 0 \ \text{ for all } \ n, m \geq 1,$$

$$\int_{-p}^{p} \cos\left(\frac{n\pi x}{p}\right) \sin\left(\frac{m\pi x}{p}\right)\, dx = 0 \quad \text{for all} \quad m, n \geq 1,$$

$$\int_{-p}^{p} \cos\left(\frac{n\pi x}{p}\right) \cos\left(\frac{m\pi x}{p}\right)\, dx = \begin{cases} 0, & m \neq n \\ p, & n = m \end{cases},$$

$$\int_{-p}^{p} \sin\left(\frac{n\pi x}{p}\right) \sin\left(\frac{m\pi x}{p}\right)\, dx = \begin{cases} 0, & m \neq n \\ p, & n = m \end{cases}.$$

Fourier Series on an interval $(-p, p)$

The orthogonality relations provide for an expansion of a function f defined on $(-p, p)$ as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{p}\right) + b_n \sin\left(\frac{n\pi x}{p}\right)\right)$$

where

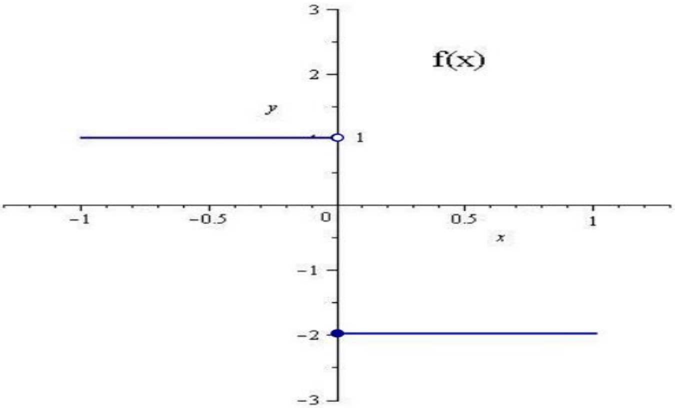
$$\begin{aligned} a_0 &= \frac{1}{p}\int_{-p}^{p} f(x)\,dx, \\ a_n &= \frac{1}{p}\int_{-p}^{p} f(x)\cos\left(\frac{n\pi x}{p}\right)\,dx, \quad \text{and} \\ b_n &= \frac{1}{p}\int_{-p}^{p} f(x)\sin\left(\frac{n\pi x}{p}\right)\,dx \end{aligned}$$

Find the Fourier series of f

$$f(x) = \begin{cases} 1, & -1 < x < 0 \\ -2, & 0 \leq x < 1 \end{cases}$$

$p = 1$

$\frac{n\pi x}{p} = \frac{n\pi x}{1} = n\pi x$



Find a_0 :

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)\,dx = \frac{1}{1}\int_{-1}^{1} f(x)\,dx$$

$$= \int_{-1}^{0} 1\,dx + \int_{0}^{1} (-2)\,dx$$

$$= x\Big|_{-1}^{0} - 2x\Big|_{0}^{1} = (0-(-1)) - 2(1-0) = 1-2 = -1$$

$$\boxed{a_0 = -1}$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\left(\frac{n\pi x}{p}\right)dx = \frac{1}{1}\int_{-1}^{1} f(x)\cos\left(\frac{n\pi x}{1}\right)dx$$

$$= \int_{-1}^{0} 1 \cos(n\pi x)\,dx + \int_{0}^{1} (-2)\cos(n\pi x)\,dx$$

$$= \frac{1}{n\pi}\sin(n\pi x)\Big|_{-1}^{0} - 2\,\frac{1}{n\pi}\sin(n\pi x)\Big|_{0}^{1}$$

$$\sin(n\pi) = \sin(-n\pi) = \sin(0) = 0$$

$$\boxed{a_n = 0 \quad \text{for } n = 1, 2, 3, \ldots}$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\left(\frac{n\pi x}{p}\right)dx = \frac{1}{1}\int_{-1}^{1} f(x)\sin\left(\frac{n\pi x}{1}\right)dx$$

$$= \int_{-1}^{0} 1 \cdot \sin(n\pi x)\,dx + \int_{0}^{1} (-2)\sin(n\pi x)\,dx$$

$$= \frac{-1}{n\pi}\cos(n\pi x)\Big|_{-1}^{0} + \frac{2}{n\pi}\cos(n\pi x)\Big|_{0}^{1}$$

$$= \frac{-1}{n\pi}\Big(\cos 0 - \cos(-n\pi)\Big) + \frac{2}{n\pi}\Big(\cos(n\pi) - \cos 0\Big)$$

$$\cos(-n\pi) = \cos(n\pi) = (-1)^n$$

$$= \frac{-1}{n\pi} + \frac{(-1)^n}{n\pi} + \frac{2(-1)^n}{n\pi} - \frac{2}{n\pi}$$

$$= \frac{3(-1)^n}{n\pi} - \frac{3}{n\pi} = \frac{3\left((-1)^n - 1\right)}{n\pi}$$

$a_0 = -1, \quad a_n = 0 \quad n \geq 1$

$$b_n = \frac{3\left((-1)^n - 1\right)}{n\pi} \qquad \frac{a_0}{2} = \frac{-1}{2}$$

$$\boxed{f(x) = \frac{-1}{2} + \sum_{n=1}^{\infty} \frac{3\left((-1)^n - 1\right)}{n\pi} \sin(n\pi x)}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{p}\right) + b_n \sin\left(\frac{n\pi x}{p}\right)\right)$$

Convergence?

The last example gave the series

$$f(x) = -\frac{1}{2} + \sum_{n=1}^{\infty} \frac{3((-1)^n - 1)}{n\pi} \sin(n\pi x).$$

This example raises an interesting question: The function f is not continuous on the interval $(-1, 1)$. However, each term in the Fourier series, and any partial sum thereof, is obviously continuous. This raises questions about properties (e.g. continuity) of the series. More to the point, we may ask: *what is the connection between f and its Fourier series at the point of discontinuity?*

This is the convergence issue mentioned earlier.

Convergence of the Series

Theorem: If f is continuous at x_0 in $(-p, p)$, then the series converges to $f(x_0)$ at that point. If f has a jump discontinuity at the point x_0 in $(-p, p)$, then the series **converges in the mean** to the average value

$$\frac{f(x_0-) + f(x_0+)}{2} \stackrel{\text{def}}{=} \frac{1}{2}\left(\lim_{x \to x_0^-} f(x) + \lim_{x \to x_0^+} f(x)\right)$$

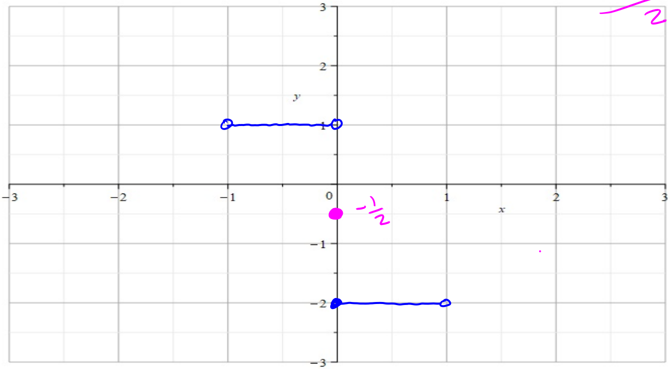
at that point.

Periodic Extension:
The series is also defined for x outside of the original domain $(-p, p)$. The extension to all real numbers is $2p$-periodic.

Convergence of the Series

$$f(x) = \begin{cases} 1, & -1 < x < 0 \\ -2, & 0 \le x < 1 \end{cases}, \quad f(x) = -\frac{1}{2} + \sum_{n=1}^{\infty} \frac{3((-1)^n - 1)}{n\pi} \sin(n\pi x).$$

$$\frac{1+(-2)}{2} = -\frac{1}{2}$$



Convergence of the Series

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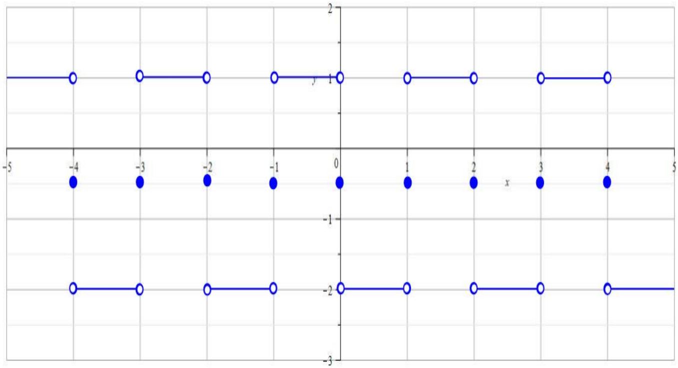

Figure: Plot of the infinite sum, the limit for the Fourier series of f.