November 29 Math 2306 sec. 52 Fall 2021

Sections 16, 17, & 18

Let's solve the following IVP using the Laplace transform.

$$y'' + 6y' + 34y = 0, \quad y(0) = 1, \quad y'(0) = 1$$

Determine $Y(s) = \mathscr{L}\{y(t)\}$ by taking the transform of the ODE, subbing in the IC, and isolating Y.

$$Y(s) = \frac{s+7}{s^2+6s+34}$$

$$y'' + 6y' + 34y = 0, \quad y(0) = 1, \quad y'(0) = 1$$

Now, do whatever algebra is necessary to write $Y(s)$ in a form from which the inverse transform can be taken.

$$Y(s) = \frac{s+7}{s^2+6s+34} = \frac{s+7-4+4}{(s+3)^2+25}$$

$$s^2+6s+34 = (s+3)^2+25$$

$$Y(s) = \frac{s+3}{(s+3)^2+5^2} + \frac{4+1-1}{(s+3)^2+5^2} = \frac{s+3}{(s+3)^2+5^2} + \frac{5}{(s+3)^2+5^2} - \frac{1}{(s+3)^2+5}$$

$$\frac{s}{s^2+5^2} \qquad \frac{5}{s^2+5^2}$$

$$Y(s) = \frac{s+3}{(s+3)^2+5^2} + \frac{4}{5}\,\frac{5}{(s+3)^2+5^2}$$

$$y'' + 6y' + 34y = 0, \quad y(0) = 1, \quad y'(0) = 1$$

Evaluate the solution to the IVP $y(t) = \mathscr{L}^{-1}\{Y(s)\}$.

$$Y(s) = \frac{s+3}{(s+3)^2 + 5^2} + \frac{4}{5}\frac{5}{(s+3)^2 + 5^2}$$

$\mathscr{L}^{-1}\left\{\frac{s}{s^2+5^2}\right\}$ $\mathscr{L}^{-1}\left\{\frac{5}{s^2+5^2}\right\}$

$\cos(5t)$ $\sin(5t)$

$\mathscr{L}^{-1}\{F(s-a)\} = e^{at}\mathscr{L}^{-1}\{F(s)\}$

$y(t) = e^{-3t}\cos(5t) + \frac{4}{5}e^{-3t}\sin(5t)$

$$y'' + 6y' + 34y = 0, \quad y(0) = 1, \quad y'(0) = 1$$

$$y(t) = e^{-3t}\cos(5t) + \frac{4}{5}e^{-3t}\sin(5t).$$

Are there any quick ways to verify that this is probably correct?

- Check to see if initial conditions are satisfied.
- Check roots of characteristic polynomial.

Example

A 1 kg mass is attached to a spring with spring constant 25 N/m. The mass is at rest at equilibrium. After $t = 2$ seconds, a unit impulse force $f(t) = \delta(t-2)$ is applied. Assuming that there is no damping, which of the following IVPs governs the displacement of the mass?

(a) $25x'' + x = 0, \quad x(0) = \delta(t-2), \quad x'(0) = 0$

(b) $x'' + 25x = 2, \quad x(0) = 0, \quad x'(0) = 0$

(c) $x'' + 25x = \delta(t-2), \quad x(0) = 0, \quad x'(0) = 0$

(d) $x'' + 2x' + 25x = 0, \quad x(0) = 0, \quad x'(0) = 0$

Example Continued...

Letting $X(s) = \mathscr{L}\{x(t)\}$, take the Laplace transform and isolate $X(s)$.

$$x'' + 25x = \delta(t-2), \quad x(0) = 0, \quad x'(0) = 0$$

$\mathscr{L}\{\delta(t-a)\} = e^{-as}$ for $a \geq 0$.

$$s^2 X(s) + 25 X(s) = e^{-2s}$$

$$(s^2 + 25) X(s) = e^{-2s}$$

$$X(s) = \frac{e^{-2s}}{s^2 + 25}$$

Example Continued...

Take the inverse Laplace transform to determine the displacement $x(t) = \mathscr{L}^{-1}\{X(s)\}$.

$$X(s) = \frac{e^{-2s}}{s^2 + 5^2}$$

$$\mathscr{L}^{-1}\{e^{-as} F(s)\} = f(t-a)\, u(t-a)$$

where $\quad f(t) = \mathscr{L}^{-1}\{F(s)\}$.

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2+5^2}\right\} = \frac{1}{5}\sin(5t)$$

$$x(t) = \frac{1}{5}\sin\left(5(t-2)\right) u(t-2)$$

$$x'' + 25x = \delta(t-2), \quad x(0) = 0, \quad x'(0) = 0$$

$$x(t) = \frac{1}{5}\sin\left(5(t-2)\right)\mathscr{U}(t-2).$$

Are there any quick ways to verify that this is probably correct?

The roots of the characteristic polynomial are ±5i, so the sin(5t) makes sense.
x(t) = 0 for t < 2, so the initial conditions are satisfied.

If the ODE has δ in it, the solution should have $\mathscr{U}$ in it.

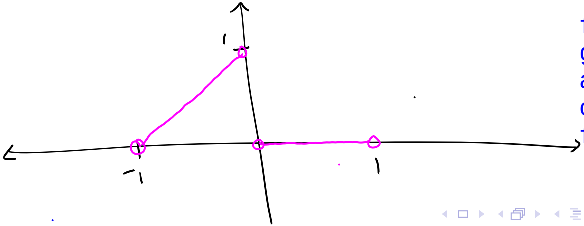
Fourier Series

Consider the function $f(x) = \begin{cases} x+1, & -1 < x < 0 \\ 0, & 0 < x < 1 \end{cases}$.

Identify p, and determine whether f is even, odd, or neither.

$p = 1$

Symmetry? no, not even or odd



This is as far as we got. The a's and b's are on the next few slides.

$$f(x) = \begin{cases} x+1, & -1 < x < 0 \\ 0, & 0 < x < 1 \end{cases}$$

$p = 1$ and this function doesn't have symmetry.

Find a_0.

$$f(x) = \begin{cases} x+1, & -1 < x < 0 \\ 0, & 0 < x < 1 \end{cases}$$

$$a_0 = \frac{1}{2}$$

Find a_n.

$$f(x) = \begin{cases} x+1, & -1 < x < 0 \\ 0, & 0 < x < 1 \end{cases}$$

$$a_n = \frac{1-(-1)^n}{n^2\pi^2}$$

Find b_n.

$$f(x) = \begin{cases} x+1, & -1 < x < 0 \\ 0, & 0 < x < 1 \end{cases}$$

$$a_0 = \frac{1}{2}, \quad a_n = \frac{1-(-1)^n}{n^2\pi^2}, \quad b_n = -\frac{1}{n\pi}$$

Write the Fourier series for f.

$$f(x) = \begin{cases} x+1, & -1 < x < 0 \\ 0, & 0 < x < 1 \end{cases}$$ Let's plot its Fourier series.

$$f(x) = \frac{1}{4} + \sum_{n=1}^{\infty} \left[\frac{1-(-1)^n}{n^2\pi^2} \cos(n\pi x) - \frac{1}{n\pi} \sin(n\pi x) \right]$$