November 29 Math 2306 sec. 52 Spring 2023

Section 16: Laplace Transforms of Derivatives and IVPs

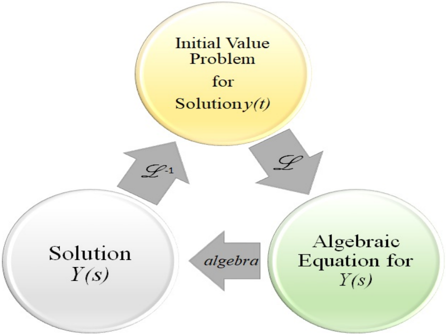

Figure: We'll use the Laplace transform as a tool for solving certain IVPs and systems of IVPs. Our use will be restricted to IVPs with **constant coefficients** and initial conditions given at $t = 0$.

The Laplace Transform of Derivatives

For $y = y(t)$ defined on $[0, \infty)$ having derivatives y', y'' and so forth, if $\mathscr{L}\{y(t)\} = Y(s)$, then

$$
\begin{aligned}
\mathscr{L}\left\{\frac{dy}{dt}\right\} &= sY(s) - y(0) \\
\mathscr{L}\left\{\frac{d^2y}{dt^2}\right\} &= s^2Y(s) - sy(0) - y'(0) \\
\mathscr{L}\left\{\frac{d^3y}{dt^3}\right\} &= s^3Y(s) - s^2y(0) - sy'(0) - y''(0) \\
&\vdots \\
\mathscr{L}\left\{\frac{d^ny}{dt^n}\right\} &= s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - y^{(n-1)}(0).
\end{aligned}
$$

Use Laplace Transforms to Solve and IVP

- Start with constant coefficient IVP with IC at $t = 0$. For example[a]

$$ay'' + by' + cy = g(t), \quad y(0) = y_0, \quad y'(0) = y_1.$$

- Let $Y(s) = \mathscr{L}\{y(t)\}$ and take the transform of both sides of the ODE using any necessary results.
- Sub in the initial conditions where they appear in the transformed equation.
- Use basic algebra to isolate the transform $Y(s)$.
- Using whatever algebra or function identities that are needed, take the inverse transform to obtain the solution

$$y(t) = \mathscr{L}^{-1}\{Y(s)\}.$$

[a] The IVP can be of any order.

Solving A System

Solve the system of initial value problems. Assume $t_0 \geq 0$ is fixed.

$$\begin{array}{rcrcrcl}x' & - & 4x & - & y & = & \delta(t-t_0), \qquad x(0)=0\\ & & 2x & + & y' & = & y, \qquad y(0)=0\end{array}$$

Let $X(s) = \mathcal{L}\{x(t)\}$ and $Y(s) = \mathcal{L}\{y(t)\}$.

$\mathcal{L}\{x' - 4x - y\} = \mathcal{L}\{\delta(t-t_0)\}$.

$\mathcal{L}\{2x + y'\} = \mathcal{L}\{y\}$.

$$sX(s) - \underset{0}{x(0)} - 4X(s) - Y(s) = e^{-t_0 s}$$

$$2X(s) + sY(s) - \underset{0}{y(0)} = Y(s)$$

$$sX(s) - 4X(s) - Y(s) = e^{-t_0 s}$$

$$2X(s) + sY(s) - Y(s) = 0$$

$$(s-4)X(s) - Y(s) = e^{-t_0 s}$$

$$2X(s) + (s-1)Y(s) = 0$$

We'll use Cramer's rule. In matrix format,

$$\begin{bmatrix} s-4 & -1 \\ 2 & s-1 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} e^{-t_0 s} \\ 0 \end{bmatrix}$$

Let $A = \begin{bmatrix} s-4 & -1 \\ 2 & s-1 \end{bmatrix}$, $A_X = \begin{bmatrix} e^{-t_0 s} & -1 \\ 0 & s-1 \end{bmatrix}$, $A_Y = \begin{bmatrix} s-4 & e^{-t_0 s} \\ 2 & 0 \end{bmatrix}$

$$\det(A) = (s-4)(s-1)+2 = s^2-5s+4+2 = s^2-5s+6 = (s-2)(s-3)$$

$$\det(A_x) = e^{-t_0 s}(s-1) - 0 = e^{-t_0}(s-1)$$

$$\det(A_y) = 0 - 2e^{-t_0 s} = -2e^{-t_0 s}$$

$$X(s) = \frac{\det(A_x)}{\det(A)} = \frac{e^{-t_0 s}(s-1)}{(s-2)(s-3)} = e^{-t_0 s}\frac{s-1}{(s-2)(s-3)}$$

$$Y(s) = \frac{\det(A_y)}{\det(A)} = \frac{-2e^{-t_0 s}}{(s-2)(s-3)} = e^{-t_0 s}\frac{-2}{(s-2)(s-3)}$$

$$\frac{s-1}{(s-2)(s-3)} = \frac{2}{s-3} - \frac{1}{s-2}$$

$$\frac{-2}{(s-2)(s-3)} = \frac{2}{s-2} - \frac{2}{s-3}$$

$$X(s) = e^{-t_0 s}\left(\frac{2}{s-3} - \frac{1}{s-2}\right)$$

$$Y(s) = e^{-t_0 s}\left(\frac{2}{s-2} - \frac{2}{s-3}\right)$$

For x, we need

$$f(t) = \mathcal{L}^{-1}\left\{\frac{2}{s-3} - \frac{1}{s-2}\right\} = 2e^{3t} - e^{2t}$$

then $x(t) = \left(2e^{3(t-t_0)} - e^{2(t-t_0)}\right)u(t-t_0)$

For y, we need

$$g(t) = \mathcal{L}^{-1}\left\{\frac{2}{s-2} - \frac{2}{s-3}\right\} = 2e^{2t} - 2e^{3t}$$

then $y(t) = \left(2e^{2(t-t_0)} - 2e^{3(t-t_0)}\right)u(t-t_0)$

The solution to the system of IVPs is

$$x(t) = \left(2e^{3(t-t_0)} - e^{2(t-t_0)}\right) u(t-t_0)$$

$$y(t) = \left(2e^{2(t-t_0)} - 2e^{3(t-t_0)}\right) u(t-t_0)$$