November 29 Math 2306 sec. 54 Fall 2021

Sections 16, 17, & 18

Let's solve the following IVP using the Laplace transform.

$$y'' + 6y' + 34y = 0$$
, $y(0) = 1$, $y'(0) = 1$

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Determine $Y(s) = \mathcal{L}{y(t)}$ by taking the transform of the ODE, subbing in the IC, and isolating *Y*.

$$Y(s) = \frac{s+7}{s^2+6s+34}$$

$$y'' + 6y' + 34y = 0$$
, $y(0) = 1$, $y'(0) = 1$

Now, do whatever algebra is necessary to write Y(s) in a form from which the inverse transform can be taken.

$$Y(s) = \frac{s+7}{s^2+6s+34} = \frac{s+7}{(s+3)^2+25}$$

$$s^{t} + 6s + 34 = (s+3)^{2} + 25$$

$$Y_{(s)} = \frac{s+3}{(s+3)^2+25} + \frac{5}{5} \frac{4}{(s+3)^2+25}$$

$$\frac{s}{s^2+5^2} = \frac{5}{s^2+5^2}$$

$$Y_{(s)} = \frac{s+3}{(s+3)^2+5^2} + \frac{4}{5} = \frac{5}{(s+3)^2+5^2}$$

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$$y'' + 6y' + 34y = 0$$
, $y(0) = 1$, $y'(0) = 1$

Evaluate the solution to the IVP $y(t) = \mathcal{L}^{-1}{Y(s)}$.

$$Y(s) = \frac{s+3}{(s+3)^2+5^2} + \frac{4}{5} \frac{5}{(s+3)^2+5^2}$$

$$y(t) = e^{3t} C_{s}(st) + \frac{4}{5} e^{3t} S_{sn}(st)$$

 $\mathcal{L}\left(e^{at}f(t)\right) = F(s-a)$

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y'' + 6y' + 34y = 0, y(0) = 1, y'(0) = 1

$$y(t) = e^{-3t}\cos(5t) + \frac{4}{5}e^{-3t}\sin(5t).$$

Are there any quick ways to verify that this is probably correct?

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Example

A 1 kg mass is attached to a spring with spring constant 25 N/m. The mass is at rest at equilibrium. After t = 2 seconds, a unit impulse force $f(t) = \delta(t-2)$ is applied. Assuming that there is no damping, which of the following IVPs governs the displacement of the mass?

(a)
$$25x''+x=0, x(0)=\delta(t-2), x'(0)=0$$

(b)
$$x''+25x=2$$
, $x(0)=0$, $x'(0)=0$

(c)
$$x''+25x = \delta(t-2), \quad x(0) = 0, \quad x'(0) = 0$$

(d) x''+2x'+25x=0, x(0)=0, x'(0)=0

Example Continued...

Letting $X(s) = \mathscr{L}{x(t)}$, take the Laplace transform and isolate X(s).

 $x'' + 25x = \delta(t-2), \quad x(0) = 0, \quad x'(0) = 0$



$$\chi(s) = \frac{e^{-xs}}{s^2 + 2s}$$

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Example Continued...

Take the inverse Laplace transform to determine the displacement $x(t) = \mathcal{L}^{-1}{X(s)}$.

$$X(s) = \frac{e^{-2s}}{s^2 + 5^2}$$

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$$\begin{aligned}
\vec{L}'\left(\vec{e}^{as}F(s)\right) &= f(t-a)U(t-a) \\
& where f(t) &= \vec{L}'(F(s)), \\
& f(t) &= \vec{L}'\left(\frac{1}{s^{2}+2s}\right) &= \frac{1}{5}\vec{d}'\left(\frac{s}{s^{2}+s^{2}}\right) &= \frac{1}{5}S(s)(st) \\
& \times (t) &= \frac{1}{5}S(s)\left(S(t-2)\right)U(t-2)
\end{aligned}$$

 $x'' + 25x = \delta(t-2), \quad x(0) = 0, \quad x'(0) = 0$ $x(t) = \frac{1}{5} \sin(5(t-2)) \mathscr{U}(t-2).$

Are there any quick ways to verify that this is probably correct?

ODEs with the delta should have U in the solution.

The roots of the characteristic polynomial are +-5i, so cos(5t) and sin(5t) are expected.

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Fourier Series

Consider the function
$$f(x) = \begin{cases} x+1, & -1 < x < 0 \\ 0, & 0 < x < 1 \end{cases}$$

Identify p, and determine whether f is even, odd, or neither.

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$$f(x) = \begin{cases} x+1, & -1 < x < 0 \\ 0, & 0 < x < 1 \end{cases}$$

p = 1 and this function doesn't have symmetry. Find a_0 .

$$\alpha_{0} = \frac{1}{1} \int_{-1}^{1} f(x) dx = \int_{-1}^{0} (x+1) dx$$
$$= \frac{1}{2}$$

$$f(x) = \begin{cases} x+1, & -1 < x < 0\\ 0, & 0 < x < 1 \end{cases}$$
$$a_0 = \frac{1}{2}$$

Find *a_n*.



$$f(x) = \begin{cases} x+1, & -1 < x < 0\\ 0, & 0 < x < 1 \\ & a_n = \frac{1-(-1)^n}{n^2 \pi^2} \end{cases}$$

Find *b_n*.



$$f(x) = \begin{cases} x+1, & -1 < x < 0\\ 0, & 0 < x < 1 \end{cases}$$
$$a_0 = \frac{1}{2}, \quad a_n = \frac{1 - (-1)^n}{n^2 \pi^2}, \quad b_n = -\frac{1}{n\pi}$$

Write the Fourier series for *f*.



 $f(x) = \left\{ \begin{array}{ll} x+1, & -1 < x < 0 \\ 0, & 0 < x < 1 \end{array} \right.$ Let's plot its Fourier series.

$$f(x) = \frac{1}{4} + \sum_{n=1}^{\infty} \left[\frac{1 - (-1)^n}{n^2 \pi^2} \cos(n\pi x) - \frac{1}{n\pi} \sin(n\pi x) \right]$$



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