November 30 Math 2306 sec. 51 Fall 2022

Section 17: Fourier Series: Trigonometric Series

Suppose f is piecewise continuous on the interval $(-p, p)$. Then we can write f as a Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{p}\right) + b_n \sin\left(\frac{n\pi x}{p}\right) \right)$$

where

$$\begin{aligned}
a_0 &= \frac{1}{p}\int_{-p}^{p} f(x)\,dx, \\
a_n &= \frac{1}{p}\int_{-p}^{p} f(x)\cos\left(\frac{n\pi x}{p}\right)\,dx, \quad \text{and} \\
b_n &= \frac{1}{p}\int_{-p}^{p} f(x)\sin\left(\frac{n\pi x}{p}\right)\,dx
\end{aligned}$$

Convergence of the Series

Theorem: If f is continuous at x_0 in $(-p, p)$, then the series converges to $f(x_0)$ at that point. If f has a jump discontinuity at the point x_0 in $(-p, p)$, then the series **converges in the mean** to the average value

$$\frac{f(x_0-) + f(x_0+)}{2} \stackrel{\text{def}}{=} \frac{1}{2}\left(\lim_{x \to x_0^-} f(x) + \lim_{x \to x_0^+} f(x)\right)$$

at that point.

Periodic Extension:
The series is also defined for x outside of the original domain $(-p, p)$. The extension to all real numbers is $2p$-periodic.

Example

$$f(x) = \begin{cases} 1, & -1 < x < 0 \\ -2, & 0 \le x < 1 \end{cases}, \quad f(x) = -\frac{1}{2} + \sum_{n=1}^{\infty} \frac{3((-1)^n - 1)}{n\pi} \sin(n\pi x).$$

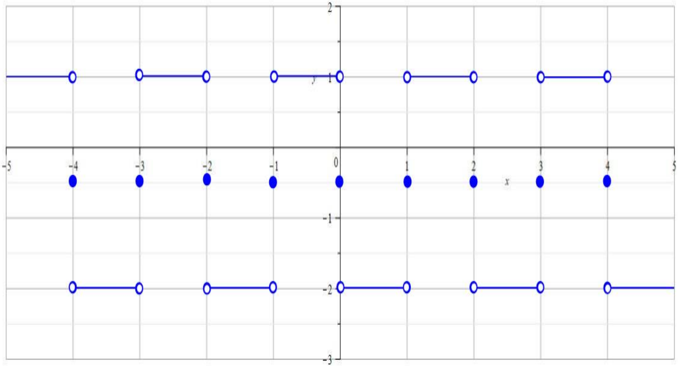


Figure: Plot of the infinite sum, the limit for the Fourier series of f.

Find the Fourier Series for $f(x) = x$, $-1 < x < 1$

$p = 1$, $\quad \frac{n\pi x}{p} = \frac{n\pi x}{1} = n\pi x$

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)\,dx = \frac{1}{1}\int_{-1}^{1} x\,dx$$

$$= \frac{x^2}{2}\Big|_{-1}^{1} = \frac{1^2}{2} - \frac{(-1)^2}{2} = 0$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\left(\frac{n\pi x}{p}\right)dx$$

$$= \frac{1}{1}\int_{-1}^{1} x\cos(n\pi x)\,dx$$

Let $u = x$, $du = dx$

$v = \frac{1}{n\pi}\sin(n\pi x)$ $dv = \cos(n\pi x)\,dx$

$$a_n = \frac{1}{n\pi}\,x\sin(n\pi x)\Big|_{-1}^{1} - \int_{-1}^{1}\frac{1}{n\pi}\sin(n\pi x)\,dx$$

0

$\sin(n\pi) = 0$

$$= \frac{1}{n^2\pi^2}\cos(n\pi x)\Big|_{-1}^{1}$$

$$= \frac{1}{n^2\pi^2}\cos(n\pi) - \frac{1}{n^2\pi^2}\cos(-n\pi)$$

$\cos(-n\pi) = \cos(n\pi)$

$$= 0$$

$a_n = 0$ for all $n \geq 1$

$$b_n = \frac{1}{p} \int_{-p}^{p} f(x) \sin\left(\frac{n\pi x}{p}\right) dx$$

$$= \int_{-1}^{1} x \sin(n\pi x) \, dx$$

$u = x$, $du = dx$,

$dv = \sin(n\pi x) \, dx$

$V = \frac{-1}{n\pi} \cos(n\pi x)$

$$= \frac{-1}{n\pi} x \cos(n\pi x) \Big|_{-1}^{1} - \int_{-1}^{1} \frac{-1}{n\pi} \cos(n\pi x) \, dx$$

$$= \frac{-1}{n\pi} 1 \cos(n\pi) - \frac{-1}{n\pi}(-1) \cos(-n\pi) + \frac{1}{n^2\pi^2} \underbrace{\cancel{\sin(n\pi x)}}_{0} \Big|_{-1}^{1}$$

$$= \frac{-1}{n\pi}\cos(n\pi) - \frac{1}{n\pi}\cos(n\pi)$$

$$= \frac{-2}{n\pi}(-1)^n = \frac{2}{n\pi}(-1)^{n+1}$$

$$a_0 = 0, \quad a_n = 0, \quad b_n = \frac{2}{n\pi}(-1)^{n+1}$$

$$\boxed{f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi}\sin(n\pi x)}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{p}\right) + b_n \sin\left(\frac{n\pi x}{p}\right) \right)$$

$f(x) = x$ on $-1 < x < 1$

Symmetry

For $f(x) = x, \quad -1 < x < 1$

$$f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi x)$$

Observation: f is an odd function. It is not surprising then that there are no nonzero constant or cosine terms (which have even symmetry) in the Fourier series for f.

The following plots show f, f plotted along with some partial sums of the series, and f along with a partial sum of its series extended outside of the original domain $(-1, 1)$.

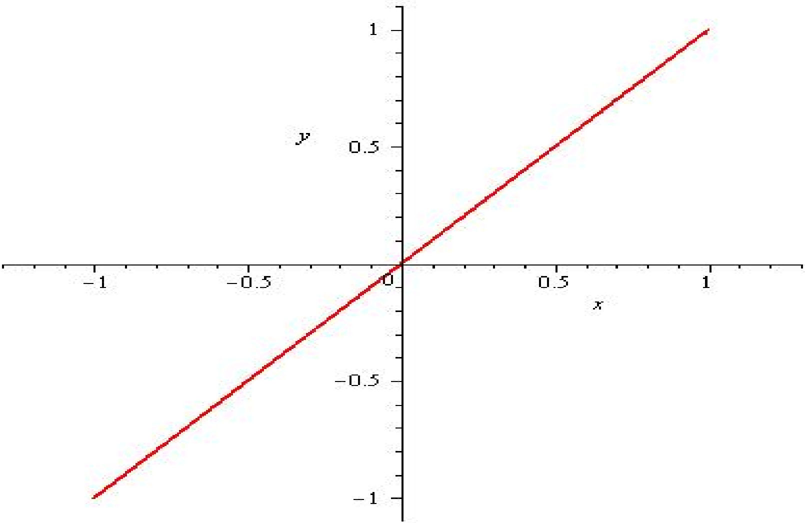


Figure: Plot of $f(x) = x$ for $-1 < x < 1$

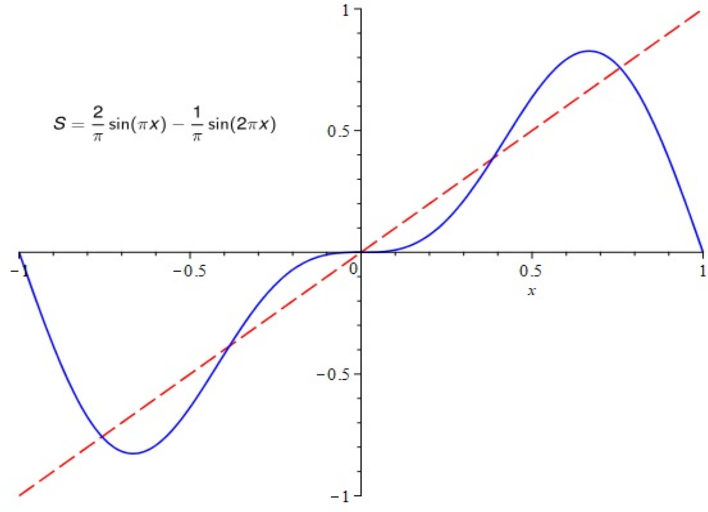


Figure: Plot of $f(x) = x$ for $-1 < x < 1$ with two terms of the Fourier series.

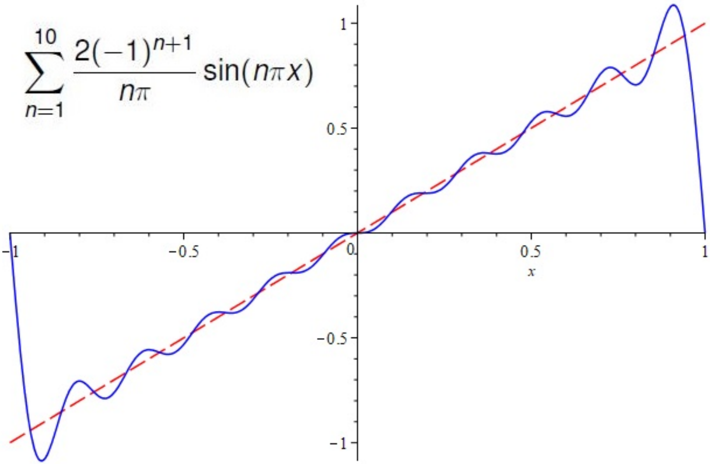
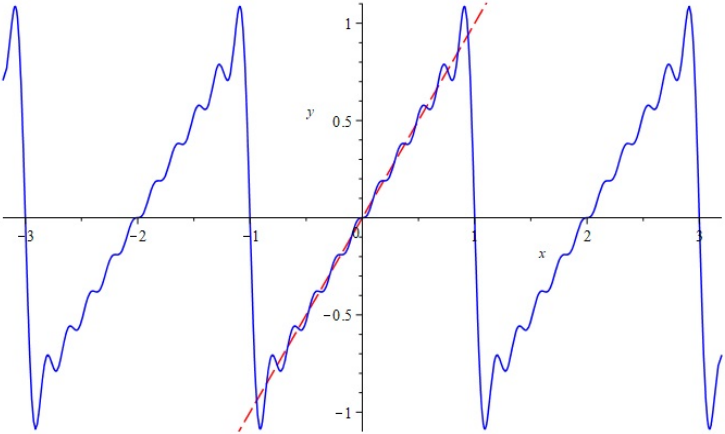
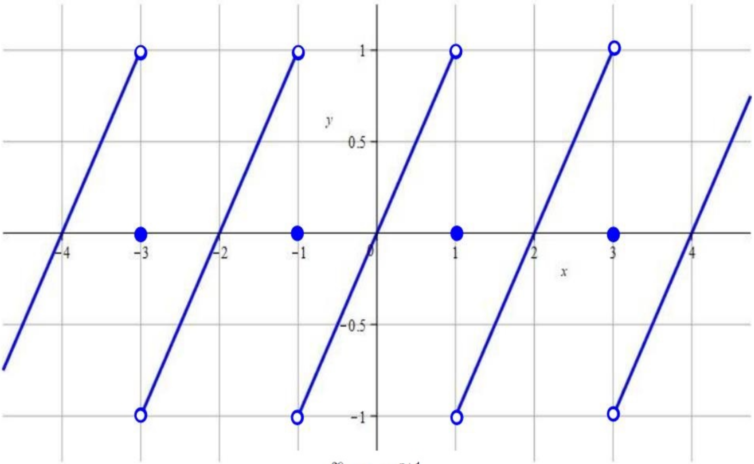


Figure: Plot of $f(x) = x$ for $-1 < x < 1$ with 10 terms of the Fourier series



$$S = \sum_{n=1}^{10} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi x)$$

Figure: Plot of $f(x) = x$ for $-1 < x < 1$ with the Fourier series plotted on $(-3, 3)$. Note that the series repeats the profile every 2 units. At the jumps, the series converges to $(-1+1)/2 = 0$.



$$S_f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi x)$$

Figure: Here is a plot of the series (what it converges to). We see the periodicity and convergence in the mean. **Note:** A plot like this is determined by our knowledge of the generating function and Fourier series, not by analyzing the series itself.

Solution of a Differential Equation

An undamped spring mass system has a mass of 2 kg attached to a spring with spring constant 128 N/m. The mass is driven by an external force $f(t) = 2t$ for $-1 < t < 1$ that is 2-periodic so that $f(t+2) = f(t)$ for all $t > 0$. Determine a particular solution x_p for the displacement for $t > 0$.

$$2x'' + 128x = f(t)$$

For $f(x) = x, \quad -1 < x < 1$

$$f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi x)$$

We have a series for our $f(t)$

$$f(t) = 2\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi t)$$

The ODE is

$$2x'' + 128x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi t)$$

$$\Rightarrow \quad x'' + 64x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi t)$$

Look for x_p in the form

$$x_p = \sum_{n=1}^{\infty} B_n \sin(n\pi t)$$

Assume that we can differentiate term by term. We'll sub this into the ODE.

$$x_p' = \sum_{n=1}^{\infty} B_n (n\pi) \cos(n\pi t)$$

$$x_p'' = \sum_{n=1}^{\infty} B_n (-n^2\pi^2) \sin(n\pi t)$$

$$x_p'' + 64 x_p = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi t)$$

$$\sum_{n=1}^{\infty} B_n (-n^2\pi^2) \sin(n\pi t) + 64 \sum_{n=1}^{\infty} B_n \sin(n\pi t) =$$

$$\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi t)$$

$$\Rightarrow \sum_{n=1}^{\infty} \left(-n^2\pi^2 B_n + 64 B_n\right) \sin(n\pi t) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi t)$$

$$\sum_{n=1}^{\infty} (64 - n^2\pi^2) B_n \sin(n\pi t) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi t)$$

Equating coefficients of $\sin(n\pi t)$ gives

$$(64-n^2\pi^2)B_n = \frac{2(-1)^{n+1}}{n\pi}$$

$$\Longrightarrow \quad B_n = \frac{2(-1)^{n+1}}{n\pi(64-n^2\pi^2)}$$

Hence

$$\boxed{x_p = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi(64-n^2\pi^2)} \sin(n\pi t)}$$