

4.6 General Vector Spaces

We've been calling R^n a **vector space** without pinning down what we really mean by this phrase. Let's remember the basic structure of R^n . We used the term **vector** to refer to an n -tuple

$$\langle x_1, x_2, \dots, x_n \rangle$$

of real numbers. And we had to consider these together with scalars, elements of R . And we had two primary operations.

vector addition: $\vec{x} + \vec{y}$

scalar multiplication: $c\vec{x}$.

Both of these operations on vectors produce vectors.

Key Properties of R^n

if $\vec{x}$, $\vec{y}$, and $\vec{z}$ are any vectors in R^n and c and d are any scalars, then

- ▶ The vector $\vec{x} + \vec{y}$ is in R^n ,
- ▶ the vector $c\vec{x}$ is in R^n ,
- ▶ $\vec{x} + \vec{y} = \vec{y} + \vec{x}$,
- ▶ $(\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$,
- ▶ there is a vector $\vec{0}_n$ such that $\vec{x} + \vec{0}_n = \vec{x}$,
- ▶ there is a vector $-\vec{x}$ such that $-\vec{x} + \vec{x} = \vec{0}_n$,
- ▶ $c(\vec{x} + \vec{y}) = c\vec{x} + c\vec{y}$,
- ▶ $(c + d)\vec{x} = c\vec{x} + d\vec{x}$,
- ▶ $c(d\vec{x}) = (cd)\vec{x} = d(c\vec{x})$, and
- ▶ $1\vec{x} = \vec{x}$.

A **real vector space** is a set, V , of objects called vectors together with two operations called **vector addition** and **scalar multiplication** that satisfy the following axioms: For each vector $\vec{x}$, $\vec{y}$, and $\vec{z}$ in V and for any scalars, c and d

1. the sum $\vec{x} + \vec{y}$ is in V , and
2. the scalar multiple $c\vec{x}$ is in V .
3. $\vec{x} + \vec{y} = \vec{y} + \vec{x}$,
4. $(\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$,
5. There is an additive identity vector in V called the zero vector denoted $\vec{0}_V$, such that $\vec{x} + \vec{0}_V = \vec{x}$ for every $\vec{x}$ in V ,
6. For each vector $\vec{x}$ in V , there is an additive inverse vector denoted $-\vec{x}$ such that $-\vec{x} + \vec{x} = \vec{0}_V$.
7. $c(\vec{x} + \vec{y}) = c\vec{x} + c\vec{y}$,
8. $(c + d)\vec{x} = c\vec{x} + d\vec{x}$,
9. $c(d\vec{x}) = (cd)\vec{x} = d(c\vec{x})$, and
10. $1\vec{x} = \vec{x}$.

Some Key Remarks

- ▶ The word **real** in the phrase **real vector space** tells us that the scalars are real number.
- ▶ The term **vector** refers to an element of a vector space. That includes things that are not real n -tuples.
- ▶ An **axiom** is a statement that is taken to be true (doesn't require proof).
- ▶ For $n \geq 1$, R^n is a real vector space.
- ▶ Concepts like **linear combination, span, linear (in)dependence, subspace, basis, dimension, and coordinate vectors** apply to abstract vector spaces.

Note: We may drop the arrow notation “ $\vec{}$ ” when working with familiar objects that are vectors, but are not usually written with arrows.

4.7 Examples of Vector Spaces

A Preliminary View of Select Vector Spaces

The Trivial Vector Space

The set $V = \{\vec{0}\}$ containing a single vector, called the **zero vector**, is a vector space. In this set, the two operations satisfy

vector addition $\vec{0} + \vec{0} = \vec{0}$, and

scalar multiplication $c\vec{0} = \vec{0}$, for scalar c .

Defining a Vector Space

It is important to note that when defining a vector space, it isn't sufficient to just identify what the vectors are. The operations of vector addition and scalar multiplication have to be defined.

A Preliminary View of Select Vector Spaces

Vector Spaces of Matrices, $M_{m \times n}$

The set $M_{m \times n}$ is the set of all $m \times n$ matrices with real entries. For vectors $A = [a_{ij}]$ and $B = [b_{ij}]$ in $M_{m \times n}$, and scalar c , the operations are defined by

vector addition $A + B = [a_{ij} + b_{ij}], \quad \text{and}$

scalar multiplication $cA = [ca_{ij}].$

These are the matrix addition and scalar multiplication from Chapter 3 (see slide 7 from June 11). We can establish that the additive identity in $M_{m \times n}$ is the $m \times n$ zero matrix $O_{m \times n}$.

A Preliminary View of Select Vector Spaces

Some Function Notation

If D is some subset of R , we use the notation

$$f : D \rightarrow R \quad \text{"}f \text{ maps } D \text{ into } R\text{"}$$

to indicate that f is a function having domain D whose outputs are real numbers. For example

- ▶ $f : [-1, 1] \rightarrow R$ defined by $f(x) = \sin^{-1}(x)$
- ▶ $g : R \rightarrow R$ defined by $g(x) = 2x^2 + 4x$
- ▶ $P : (0, \infty) \rightarrow R$ defined by $P(x) = \frac{1}{x}$
- ▶ $N : (-\infty, 0) \rightarrow R$ defined by $N(x) = \frac{1}{x}$

Note that two function f and g are **equal** if and only if they have the same domain D and $f(x) = g(x)$ for each x in D . For example, even though $P(x) = \frac{1}{x}$ and $N(x) = \frac{1}{x}$ above, $P \neq N$.

A Preliminary View of Select Vector Spaces

Function Spaces $F(D)$

For a given subset D of R , $F(D) = \{f \mid f : D \rightarrow R\}$. The¹ additive identity vector

$$z(x) = 0, \quad \text{for each } x \in D.$$

For two elements f and g in $F(D)$ and scalar c ,

vector addition $(f + g)(x) = f(x) + g(x), \quad \text{and}$

scalar multiplication $(cf)(x) = cf(x).$

¹I'm using the article **the**, but it's not immediately clear that there is only one additive identity vector.

A Preliminary View of Select Vector Spaces

The Vector Space R^∞

The vectors in R^∞ are the infinite sequences of real number,

$$\vec{a} = \langle a_1, a_2, a_3, \dots \rangle$$

where $a_i \in R$. For $\vec{a}$ and $\vec{b} = \langle b_1, b_2, b_3, \dots \rangle$ in R^∞ and scalar c ,

vector addition $\vec{a} + \vec{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3, \dots \rangle$, and

scalar multiplication $c\vec{a} = \langle ca_1, ca_2, ca_3, \dots \rangle$.

4.6 General Vector Spaces

Back to Concepts & Results on Vector Spaces

Theorem

Suppose that V is a vector space. Then

1. There is only one additive identity vector in V (i.e., the zero vector of V is unique).
2. Each vector in V has only one additive inverse (i.e., the additive inverse of any vector in V is unique).
3. If $\vec{x}$ is any vector in V , then $0\vec{x} = \vec{0}_V$.
4. If c is any scalar, then $c\vec{0}_V = \vec{0}_V$.

Note: Each statement can be proved by appealing to the vector space axioms without making any assumptions about what sort of objects the vectors actually are.

Let's prove 3. If $\vec{x}$ is any vector in V , then $0\vec{x} = \vec{0}_V$.

We will only use these vector space axioms.

For every $\vec{x}, \vec{y}, \vec{z}$ in V and scalars c and d

1. the sum $\vec{x} + \vec{y}$ is in V , and
2. the scalar multiple $c\vec{x}$ is in V .
3. $\vec{x} + \vec{y} = \vec{y} + \vec{x}$,
4. $(\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$,
5. There is an additive identity vector in V called the zero vector denoted $\vec{0}_V$, such that $\vec{x} + \vec{0}_V = \vec{x}$ for every $\vec{x}$ in V ,
6. For each vector $\vec{x}$ in V , there is an additive inverse vector denoted $-\vec{x}$ such that $-\vec{x} + \vec{x} = \vec{0}_V$.
7. $c(\vec{x} + \vec{y}) = c\vec{x} + c\vec{y}$,
8. $(c + d)\vec{x} = c\vec{x} + d\vec{x}$,
9. $c(d\vec{x}) = (cd)\vec{x} = d(c\vec{x})$, and
10. $1\vec{x} = \vec{x}$.

We'll need 8., 2., 6., 4., and 5.

Proof of 3.: If $\vec{x}$ is any vector in V , then $0\vec{x} = \vec{0}_V$.

Linear Combination

Let $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ be a set of one or more ($k \geq 1$) vectors in a vector space V . A **linear combination** of these vectors is any vector of the form

$$x_1 \vec{v}_1 + x_2 \vec{v}_2 + \dots + x_k \vec{v}_k,$$

where $x_1, x_2, \dots, x_k$ are scalars. The coefficients, $x_1, x_2, \dots, x_k$, are often called the **weights**.

Span

Let $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ be a set of one or more ($k \geq 1$) vectors in a vector space V . The set of all possible linear combinations of the vectors in S is called the **span** of S . It is denoted by $\text{Span}(S)$ or by $\text{Span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$.

Example

Consider the vector space $M_{2 \times 2}$ of 2×2 matrices with real entries.

$$\text{Let } E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

and let $D = \text{Span}\{E_{11}, E_{22}\}$. Describe the subset D of $M_{2 \times 2}$.

Linear Independence/Dependence

Let V be a vector space. The collection of vectors $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ in V is said to be **linearly independent** if the homogeneous equation

$$x_1 \vec{v}_1 + x_2 \vec{v}_2 + \dots + x_k \vec{v}_k = \vec{0}_V \quad (1)$$

has only the trivial solution, $x_1 = x_2 = \dots = x_k = 0$. A set of vectors that is not linearly independent is called **linearly dependent**.

If a set of vectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ in V is linearly dependent, an equation of the form

$$x_1 \vec{v}_1 + x_2 \vec{v}_2 + \dots + x_k \vec{v}_k = \vec{0}_V$$

with at least one coefficient $x_i \neq 0$ is called a **linear dependence relation**.

Example: $E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Show that the set $\{E_{11}, E_{22}\}$ is linearly independent² in $M_{2 \times 2}$.

²Recall that the additive identity is the 2×2 zero matrix.

Subspace

Let V be a real vector space. A **subspace** of V is a nonempty set, S , of vectors in V such that

- ▶ for every $\vec{x}$ and $\vec{y}$ in S , $\vec{x} + \vec{y}$ is in S , and
- ▶ for each $\vec{x}$ in S and scalar c , $c\vec{x}$ is in S .

Remark: A subspace of a vector space must contain the zero vector $\vec{0}_V$. And a span is always a subspace.

Basis

Let S be a subspace of a vector space V , and let $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_k\}$ be a subset of vectors in S . $\mathcal{B}$ is a **basis** of S provided

- ▶ $\text{Span}(\mathcal{B}) = S$
- ▶ $\mathcal{B}$ is linearly independent.

Example

Consider the vector space $F(R)$, and consider the vectors $f(x) = \sin^2(x)$ and $g(x) = \cos(2x)$ in $F(R)$.

1. Describe the elements of $S = \text{Span}\{f, g\}$.
2. Show that $h(x) = 1$ is in $\text{Span}\{f, g\}$.
3. Use this to show that the set $\{f, g, h\}$ is linearly dependent.

Example

Consider the subset T of $M_{2 \times 2}$ given by

$$T = \left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \mid a, b, c \in R \right\}.$$

Show that T is a subspace³ of $M_{2 \times 2}$.

³There is a special name for these matrices. They're called *trace free* matrices. The trace of a square matrix is the sum of its diagonal entries.

It can be proved that if a subspace S of a vector space V has a basis with k vectors in it ($1 \leq k < \infty$) then every basis of S has exactly k vectors in it.

Dimension

Let S be a subspace of a vector space V . We define the **dimension** of S as follows:

- ▶ If $S = \{\vec{0}_V\}$, then we define $\dim(S) = 0$.
- ▶ If S has a basis consisting of k vectors, where $k < \infty$, then we define $\dim(S) = k$.
- ▶ If S is not spanned by any finite set of vectors, then we say that S is infinite dimensional.

The spaces R^∞ and $F(D)$ are examples of infinite dimensional vector spaces. Bases for such vector spaces are known to exist but are not particularly useful. We can readily construct finite dimensional subspaces of infinite dimensional vector spaces.

Example $M_{2 \times 2}$

The set $\mathcal{E} = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ is a basis for $M_{2 \times 2}$

$$E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

(Proof is left as an exercise.)

It follows that $\dim(M_{2 \times 2}) = \underline{\hspace{2cm}}$.

Example

Find a basis for the subspace $T = \left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \mid a, b, c \in R \right\}$ of $M_{2 \times 2}$, and determine $\dim(T)$.

