November 3 Math 3260 sec. 53 Fall 2025

4.6 General Vector Spaces

We've been calling R^n a **vector space** without pinning down what we really mean by this phrase. Let's remember the basic structure of R^n. We used the term **vector** to refer to an n-tuple

$$\langle x_1, x_2, \dots, x_n \rangle$$

of real numbers. And we had to consider these together with scalars, elements of R. And we had two primary operations.

$$\text{vector addition:} \quad \vec{x} + \vec{y}$$

$$\text{scalar multiplication:} \quad c\vec{x}.$$

Both of these operations on vectors produce vectors.

Key Properties of R^n

if $\vec{x}$, $\vec{y}$, and $\vec{z}$ are any vectors in R^n and c and d are any scalars, then

- The vector $\vec{x} + \vec{y}$ is in R^n,
- the vector $c\vec{x}$ is in R^n,
- $\vec{x} + \vec{y} = \vec{y} + \vec{x}$,
- $(\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$,
- there is a vector $\vec{0}_n$ such that $\vec{x} + \vec{0}_n = \vec{x}$,
- there is a vector $-\vec{x}$ such that $-\vec{x} + \vec{x} = \vec{0}_n$,
- $c(\vec{x} + \vec{y}) = c\vec{x} + c\vec{y}$,
- $(c + d)\vec{x} = c\vec{x} + d\vec{x}$,
- $c(d\vec{x}) = (cd)\vec{x} = d(c\vec{x})$, and
- $1\vec{x} = \vec{x}$.

A **real vector space** is a set, V, of objects called vectors together with two operations called **vector addition** and **scalar multiplication** that satisfy the following axioms: For each vector $\vec{x}$, $\vec{y}$, and $\vec{z}$ in V and for any scalars, c and d

1. the sum $\vec{x}+\vec{y}$ is in V, and
2. the scalar multiple $c\vec{x}$ is in V.
3. $\vec{x}+\vec{y}=\vec{y}+\vec{x}$,
4. $(\vec{x}+\vec{y})+\vec{z}=\vec{x}+(\vec{y}+\vec{z})$,
5. There is an additive identity vector in V called the zero vector denoted $\vec{0}_V$, such that $\vec{x}+\vec{0}_V=\vec{x}$ for every $\vec{x}$ in V,
6. For each vector $\vec{x}$ in V, there is an additive inverse vector denoted $-\vec{x}$ such that $-\vec{x}+\vec{x}=\vec{0}_V$.
7. $c(\vec{x}+\vec{y})=c\vec{x}+c\vec{y}$,
8. $(c+d)\vec{x}=c\vec{x}+d\vec{x}$,
9. $c(d\vec{x})=(cd)\vec{x}=d(c\vec{x})$, and
10. $1\vec{x}=\vec{x}$.

Some Key Remarks

- The word **real** in the phrase **real vector space** tells us that the scalars are real number.
- The term **vector** refers to an element of a vector space. That includes things that are not real n-tuples.
- An **axiom** is a statement that is taken to be true (doesn't require proof).
- For $n \geq 1$, R^n is a real vector space.
- Concepts like **linear combination, span, linear (in)dependence, subspace, basis, dimension, and coordinate vectors** apply to abstract vector spaces.

Note: We may drop the arrow notation "$\vec{}$" when working with familiar objects that are vectors, but are not usually written with arrows.

The Trivial Vector Space

The set $V = \left\{\vec{0}\right\}$ containing a single vector, called the **zero vector**, is a vector space. In this set, the two operations satisfy

$$\textbf{vector addition} \quad \vec{0}+\vec{0}=\vec{0}, \quad \text{and}$$

$$\textbf{scalar multiplication} \quad c\vec{0}=\vec{0}, \quad \text{for scalar } c.$$

Defining a Vector Space

It is important to note that when defining a vector space, it isn't sufficient to just identify what the vectors are. The operations of vector addition and scalar multiplication have to be defined.

A Preliminary View of Select Vector Spaces

Vector Spaces of Matrices, $M_{m\times n}$

The set $M_{m\times n}$ is the set of all $m \times n$ matrices with real entries. For vectors $A = [a_{ij}]$ and $B = [b_{ij}]$ in $M_{m\times n}$, and scalar c, the operations are defined by

$$\textbf{vector addition} \quad A + B = [a_{ij} + b_{ij}], \quad \text{and}$$

$$\textbf{scalar multiplication} \quad cA = [ca_{ij}].$$

These are the matrix addition and scalar multiplication from Chapter 3 (see slide 13 from Sept. 19). We can establish that the additive identity in $M_{m\times n}$ is the $m \times n$ zero matrix $O_{m\times n}$.

A Preliminary View of Select Vector Spaces

Some Function Notation

If D is some subset of R, we use the notation

$$f : D \to R \quad \textit{“f maps D into R”}$$

to indicate that f is a function having domain D whose outputs are real numbers. For example

- $f : [-1, 1] \to R$ defined by $f(x) = \sin^{-1}(x)$
- $g : R \to R$ defined by $g(x) = 2x^2 + 4x$
- $P : (0, \infty) \to R$ defined by $P(x) = \frac{1}{x}$
- $N : (-\infty, 0) \to R$ defined by $N(x) = \frac{1}{x}$

Note that two function f and g are **equal** if and only if they have the same domain D and $f(x) = g(x)$ for each x in D. For example, even though $P(x) = \frac{1}{x}$ and $N(x) = \frac{1}{x}$ above, $P \neq N$.

A Preliminary View of Select Vector Spaces

Function Spaces $F(D)$

For a given subset D of R, $F(D) = \{f \mid f : D \to R\}$. The[1] additive identity vector

$$z(x) = 0, \quad \text{for each } x \in D.$$

For two elements f and g in $F(D)$ and scalar c,

$$\textbf{vector addition} \quad (f + g)(x) = f(x) + g(x), \quad \text{and}$$

$$\textbf{scalar multiplication} \quad (cf)(x) = cf(x).$$

[1] I'm using the article **the**, but it's not immediately clear that there is only one additive identity vector.

A Preliminary View of Select Vector Spaces

The Vector Space R^∞

an alternative notation

The vectors in R^∞ are the infinite sequences of real number,

$$\vec{a} = \langle a_1, a_2, a_3, \dots \rangle \qquad \{a_n\}_{n=1}^{\infty}$$

where $a_i \in R$. For $\vec{a}$ and $\vec{b} = \langle b_1, b_2, b_3, \dots \rangle$ in R^∞ and scalar c,

vector addition $\quad \vec{a} + \vec{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3, \dots \rangle, \quad$ and

scalar multiplication $\quad c\vec{a} = \langle ca_1, ca_2, ca_3, \dots \rangle.$

4.6 General Vector Spaces

Back to Concepts & Results on Vector Spaces

Theorem

Suppose that V is a vector space. Then

1. There is only one additive identity vector in V (i.e., the zero vector of V is unique).
2. Each vector in V has only one additive inverse (i.e., the additive inverse of any vector in V is unique).
3. If $\vec{x}$ is any vector in V, then $0\vec{x} = \vec{0}_V$.
4. If c is any scalar, then $c\vec{0}_V = \vec{0}_V$.

Note: Each statement can be proved by appealing to the vector space axioms without making any assumptions about what sort of objects the vectors actually are.

Let's prove 3. If $\vec{x}$ is any vector in V, then $0\vec{x} = \vec{0}_V$.

We will only use these vector space axioms.

For every $\vec{x}, \vec{y}, \vec{z}$ in V and scalars c and d

1. the sum $\vec{x} + \vec{y}$ is in V, and
2. the scalar multiple $c\vec{x}$ is in V.
3. $\vec{x} + \vec{y} = \vec{y} + \vec{x}$,
4. $(\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$,
5. There is an additive identity vector in V called the zero vector denoted $\vec{0}_V$, such that $\vec{x} + \vec{0}_V = \vec{x}$ for every $\vec{x}$ in V,
6. For each vector $\vec{x}$ in V, there is an additive inverse vector denoted $-\vec{x}$ such that $-\vec{x} + \vec{x} = \vec{0}_V$.
7. $c(\vec{x} + \vec{y}) = c\vec{x} + c\vec{y}$,
8. $(c + d)\vec{x} = c\vec{x} + d\vec{x}$,
9. $c(d\vec{x}) = (cd)\vec{x} = d(c\vec{x})$, and
10. $1\vec{x} = \vec{x}$.

We'll need 8., 2., 6., 4., and 5.

Proof of 3.: If $\vec{x}$ is any vector in V, then $0\vec{x} = \vec{0}_V$.

Let $\vec{x}$ be any vector in V. Recall that the real scaler zero satisfies $0+0=0$.

So,
$$0\vec{x} = (0+0)\vec{x}$$
$$= 0\vec{x} + 0\vec{x} \quad \text{by axiom 8.}$$

By axiom 2, $0\vec{x}$ is a vector in V. So by axiom 6, there is an additive inverse $-0\vec{x}$. Add this to both sides of the above equation.

$$-0\vec{x} + 0\vec{x} = -0\vec{x} + 0\vec{x} + 0\vec{x}$$

By axiom 6, the left is $\vec{0}_V$. By axiom 4, the right is $(-0\vec{x} + 0\vec{x}) + 0\vec{x}$.

$$\vec{0}_V = (-0\vec{x} + 0\vec{x}) + 0\vec{x}$$

$$\vec{0}_V = \vec{0}_V + 0\vec{x} \quad \text{by axiom 6.}$$

$$\vec{0}_V = 0\vec{x} \quad \text{by axiom 5.}$$

as required.

Linear Combination

Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ be a set of one or more ($k \geq 1$) vectors in a vector space V. A **linear combination** of these vectors is any vector of the form

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_k\vec{v}_k,$$

where $x_1,, x_2, \ldots, x_k$ are scalars. The coefficients, $x_1, x_2, \ldots, x_k$, are often called the **weights**.

Span

Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ be a set of one or more ($k \geq 1$) vectors in a vector space V. The set of all possible linear combinations of the vectors in S is called the **span** of S. It is denoted by $\mathrm{Span}(S)$ or by $\mathrm{Span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$.

Example

Consider the vector space $M_{2\times 2}$ of 2×2 matrices with real entries.

$$\text{Let} \quad E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

and let $D = \mathrm{Span}\{E_{11}, E_{22}\}$. Describe the subset D of $M_{2\times 2}$.

Suppose A is in D. Then

$$A = x_1 E_{11} + x_2 E_{22} \quad \text{for some scalars } x_1, x_2$$

$$= x_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix}.$$

So the vectors in D are 2×2 matrices with any real numbers on the main diagonal and zero off the main diagonal.

Linear Independence/Dependence

Let V be a vector space. The collection of vectors $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ in V is said to be **linearly independent** if the homogeneous equation

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_k\vec{v}_k = \vec{0}_V \tag{1}$$

has only the trivial solution, $x_1 = x_2 = \cdots = x_k = 0$. A set of vectors that is not linearly independent is called **linearly dependent.**

If a set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ in V is linearly dependent, an equation of the form

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_k\vec{v}_k = \vec{0}_V$$

with at least one coefficient $x_i \neq 0$ is called a **linear dependence relation**.

Example: $E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Show that the set $\{E_{11}, E_{22}\}$ is linearly independent[2] in $M_{2\times 2}$.

Consider $x_1 E_{11} + x_2 E_{22} = O_{2\times 2}$

$$\begin{bmatrix} x_1 & 0 \\ 0 & x_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow x_1 = 0 \text{ and } x_2 = 0$$

Since both x_1 and x_2 must be zero, $\{E_{11}, E_{22}\}$ is linearly independent.

[2] Recall that the additive identity is the 2×2 zero matrix.

Subspace

Let V be a real vector space. A **subspace** of V is a nonempty set, S, of vectors in V such that

- for every $\vec{x}$ and $\vec{y}$ in S, $\vec{x}+\vec{y}$ is in S, and
- for each $\vec{x}$ in S and scalar c, $c\vec{x}$ is in S.

Remark: A subspace of a vector space must contain the zero vector $\vec{0}_V$. And a span is always a subspace.

Basis

Let S be a subspace of a vector space V, and let $\mathcal{B}=\{\vec{v}_1,\ldots,\vec{v}_k\}$ be a subset of vectors in S. $\mathcal{B}$ is a **basis** of S provided

- $\mathrm{Span}(\mathcal{B})=S$
- $\mathcal{B}$ is linearly independent.

Example

Consider the vector space $F(R)$, and consider the vectors $f(x) = \sin^2(x)$ and $g(x) = \cos(2x)$ in $F(R)$.

1. Describe the elements of $S = \mathrm{Span}\{f, g\}$.
2. Show that $h(x) = 1$ is in $\mathrm{Span}\{f, g\}$.
3. Use this to show that the set $\{f, g, h\}$ is linearly dependent.

$F(\mathbb{R}) = \{ f \mid f : \mathbb{R} \to \mathbb{R} \}$.

1. Let p be in S. Then

$p(x) = c_1 f(x) + c_2 g(x)$ for some scalars c_1 and c_2.

$p(x) = c_1 \sin^2(x) + c_2 \cos(2x)$

2. Recall from trig $\cos(2x) = 1 - 2\sin^2(x)$

$\Rightarrow \quad 1 = 2\sin^2(x) + \cos(2x)$ for all real x.

So $h(x) = 2f(x) + 1g(x)$

3. From 2., $2f(x) + 1g(x) - 1h(x) = 0 = z(x)$

This is a linear dependence relation showing that $\{f, g, h\}$ is linearly dependent.