November 4 Math 2306 sec. 53 Fall 2024

Section 13: The Laplace Transform

Definition: The Laplace Transform

Let $f(t)$ be piecewise continuous on $[0, \infty)$. The Laplace transform of f, denoted $\mathscr{L}\{f(t)\}$ is given by.

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt. \quad = F(s)$$

We will often use the upper case/lower case convention that $\mathscr{L}\{f(t)\}$ will be represented by $F(s)$. The domain of the transformation $F(s)$ is the set of all s such that the integral is convergent.

A piecewise defined function

Find the Laplace transform of f defined by

$$f(t) = \begin{cases} 2t, & 0 \le t < 10 \\ 0, & t \ge 10 \end{cases}$$

We were in the process of evaluating this transform.

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = \int_0^{10} 2te^{-st}\, dt + \int_{10}^\infty 0 \cdot e^{-st}\, dt$$

So we ended up with

$$F(s) = \mathscr{L}\{f(t)\} = \int_0^{10} 2te^{-st}\, dt$$

When $s = 0$, $F(0) = \int_0^{10} 2t\, dt = t^2 \Big|_0^{10} = 10^2 - 0 = 100$

For $s \neq 0$

$u = 2t \quad du = 2\,dt$

$$F(s) = \int_0^{10} 2t e^{-st}\, dt$$

$v = \frac{-1}{s} e^{-st} \quad dv = e^{-st}\,dt$

$$= \frac{-2t}{s} e^{-st} \Big|_0^{10} - \int_0^{10} \frac{-2}{s} e^{-st}\, dt$$

$$= \frac{-20}{s} e^{-10s} - 0 + \frac{2}{s} \left[\frac{-1}{s} e^{-st} \Big|_0^{10} \right.$$

$$= \frac{-20}{s} e^{-10s} + \frac{2}{s} \left(\frac{-1}{s} e^{-10s} - \frac{-1}{s} e^{0} \right)$$

$$= \frac{-20}{s} e^{-10s} - \frac{2}{s^2} e^{-10s} + \frac{2}{s^2}$$

$$F(s) = \begin{cases} 100, & s = 0 \\ \frac{2}{s^2} - \frac{2}{s^2} e^{-10s} - \frac{20}{s} e^{-10s}, & s \neq 0 \end{cases}$$

It's not obvious, but this is continuous. That is, if you take the limit as s tends to 0 of the bottom piece, you do get 100.

Computing Laplace Transforms

Despite the definition, Laplace transforms are rarely evaluated by actually integrating. Transforms of common functions (and some *not-so-common*) are extensively cataloged. Tables of transforms are used in practice.

Remark: Googling *table of Laplace transforms* will yield thousands of free webpages and pdfs. The table I'll provide during exams is posted in D2L (and the course page, and the workbook).

Figure: Table of Laplace transforms t-shirt.

A Small Table of Laplace Transforms

Some basic results include:

- $\mathscr{L}\{\alpha f(t) + \beta g(t)\} = \alpha F(s) + \beta G(s)$
- $\mathscr{L}\{1\} = \frac{1}{s}, \quad s > 0$
- $\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad s > 0$ for $n = 1, 2, \ldots$
- $\mathscr{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a$
- $\mathscr{L}\{\cos kt\} = \frac{s}{s^2+k^2}, \quad s > 0$
- $\mathscr{L}\{\sin kt\} = \frac{k}{s^2+k^2}, \quad s > 0$

Evaluate the Laplace transform $\mathscr{L}\{f(t)\}$ if

$$\mathscr{L}\{\cos kt\} = \frac{s}{s^2+k^2}, \quad s > 0$$

(a) $f(t) = \cos(\pi t)$

Here, $k = \pi$

$$\mathscr{L}\{\cos(\pi t)\} = \frac{s}{s^2 + \pi^2}$$

Evaluate the Laplace transform $\mathscr{L}\{f(t)\}$ if

$\mathscr{L}\{1\} = \frac{1}{s}, \quad s > 0$

$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad s > 0$ for $n = 1, 2, \dots$

$\mathscr{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a$

(b) $f(t) = 2t^4 - e^{-5t} + 3$

$$\mathcal{L}\{2t^4 - e^{-5t} + 3\} = 2\mathcal{L}\{t^4\} - \mathcal{L}\{e^{-5t}\} + 3\mathcal{L}\{1\}$$

$$= 2\left(\frac{4!}{s^{4+1}}\right) - \frac{1}{s-(-5)} + 3\left(\frac{1}{s}\right)$$

$$= \frac{2(4!)}{s^5} - \frac{1}{s+5} + \frac{3}{s}$$

Helpful Tip

If it's not immediately obvious how to evaluate $\mathscr{L}\{f(t)\}$, it's almost always helpful to consider the question

How would I evaluate $\displaystyle\int f(t)\,dt$?

The algebra and function identities used to evaluate integrals are *usually* helpful for evaluating Laplace transforms.

Evaluate the Laplace transform $\mathscr{L}\{f(t)\}$ if

(c) $f(t) = (2-t)^2 = 4 - 4t + t^2$

$\mathscr{L}\{1\} = \frac{1}{s}, \quad s > 0$

$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad s > 0$ for $n = 1, 2, \ldots$

$\mathscr{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a$

$$\mathscr{L}\{(2-t)^2\} = \mathscr{L}\{4 - 4t + t^2\}$$

$$= 4\mathscr{L}\{1\} - 4\mathscr{L}\{t\} + \mathscr{L}\{t^2\}$$

$$= \frac{4}{s} - 4\frac{1!}{s^{1+1}} + \frac{2!}{s^{2+1}}$$

$$= \frac{4}{s} - \frac{4}{s^2} + \frac{2}{s^3}$$

Examples: Evaluate

(d) $\mathscr{L}\{\sin^2 5t\}$

$\mathscr{L}\{\cos kt\} = \frac{s}{s^2+k^2}, \quad s > 0$

$\mathscr{L}\{\sin kt\} = \frac{k}{s^2+k^2}, \quad s > 0$

$$\mathscr{L}\{\sin^2(5t)\} = \mathscr{L}\{\tfrac{1}{2} - \tfrac{1}{2}\cos(10t)\}$$

$$= \frac{1}{2}\mathscr{L}\{1\} - \frac{1}{2}\mathscr{L}\{\cos(10t)\}$$

$$= \frac{1}{2}\left(\frac{1}{s}\right) - \frac{1}{2}\,\frac{s}{s^2+10^2}$$

$$= \frac{\frac{1}{2}}{s} - \frac{1}{2}\,\frac{s}{s^2+100}$$

$$\int \sin^2(5t)\,dt$$

$$\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$$

Sufficient Conditions for Existence of $\mathscr{L}\{f(t)\}$

Definition: Exponential Order

Let $c > 0$. A function f defined on $[0, \infty)$ is said to be of *exponential order c* provided there exists positive constants M and T such that $|f(t)| < Me^{ct}$ for all $t > T$.

Being *of exponential order c* means that f doesn't blow up at infinity any faster than e^{ct}.

Definition:Piecewise Continuous

A function f is said to be *piecewise continuous* on an interval $[a, b]$ if f has at most finitely many jump discontinuities on $[a, b]$ and is continuous between each such jump.

Sufficient Conditions for Existence of $\mathscr{L}\{f(t)\}$

Theorem:

If f is piecewise continuous on $[0, \infty)$ and of exponential order c for some real number c, then f has a Laplace transform valid for $s > c$.

An example of a function that is NOT of exponential order for any c is $f(t) = e^{t^2}$. Note that

$$f(t) = e^{t^2} = \left(e^t\right)^t \quad \Longrightarrow \quad |f(t)| > e^{ct} \quad \text{whenever} \quad t > c.$$

This is a function that doesn't have a Laplace transform. We won't be dealing with this type of function here.