November 5 Math 3260 sec. 51 Fall 2025

4.6 General Vector Spaces

We defined a general **real vector space** and various concepts including

- linear combination and span,
- linear dependence/independence,
- subspaces, and
- bases.

We also saw some examples of vector spaces such as

$$R^n, \quad M_{m\times n}, \quad F(D), \quad \text{and} \quad R^\infty$$

A **real vector space** is a set, V, of objects called vectors together with two operations called **vector addition** and **scalar multiplication** that satisfy the following axioms: For each vector $\vec{x}$, $\vec{y}$, and $\vec{z}$ in V and for any scalars, c and d

1. the sum $\vec{x} + \vec{y}$ is in V, and
2. the scalar multiple $c\vec{x}$ is in V.
3. $\vec{x} + \vec{y} = \vec{y} + \vec{x}$,
4. $(\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$,
5. There is an additive identity vector in V called the zero vector denoted $\vec{0}_V$, such that $\vec{x} + \vec{0}_V = \vec{x}$ for every $\vec{x}$ in V,
6. For each vector $\vec{x}$ in V, there is an additive inverse vector denoted $-\vec{x}$ such that $-\vec{x} + \vec{x} = \vec{0}_V$.
7. $c(\vec{x} + \vec{y}) = c\vec{x} + c\vec{y}$,
8. $(c + d)\vec{x} = c\vec{x} + d\vec{x}$,
9. $c(d\vec{x}) = (cd)\vec{x} = d(c\vec{x})$, and
10. $1\vec{x} = \vec{x}$.

Subspace

Let V be a real vector space. A **subspace** of V is a nonempty set, S, of vectors in V such that

- for every $\vec{x}$ and $\vec{y}$ in S, $\vec{x}+\vec{y}$ is in S, and
- for each $\vec{x}$ in S and scalar c, $c\vec{x}$ is in S.

Remark: A subspace of a vector space must contain the zero vector $\vec{0}_V$. And a span is always a subspace.

Basis

Let S be a subspace of a vector space V, and let $\mathcal{B}=\{\vec{v}_1,\ldots,\vec{v}_k\}$ be a subset of vectors in S. $\mathcal{B}$ is a **basis** of S provided

- $\mathrm{Span}(\mathcal{B})=S$
- $\mathcal{B}$ is linearly independent.

Example

Consider the subset T of $M_{2\times 2}$ given by

$$T = \left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \;\middle|\; a, b, c \in R \right\}.$$

Show that T is a subspace[1] of $M_{2\times 2}$.

We can show that T is nonempty and closed under vector addition and scalar multiplication. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is in T, so T is nonempty.

Let A and B be any elements of T.

Then $A = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ and $B = \begin{bmatrix} d & e \\ f & -d \end{bmatrix}$ for some scalars $a, b, \ldots, f$

[1] There is a special name for these matrices. They're called *trace free* matrices. The trace of a square matrix is the sum of its diagonal entries.

$A+B = \begin{bmatrix} a+d & b+e \\ c+f & -a-d \end{bmatrix}$. Adding the diagonal

entries, $a+d+(-a-d) = a+d-a-d = 0$.

So $A+B$ is in T. T is closed under

vector addition. Let k be any scalar.

$kA = \begin{bmatrix} ka & kb \\ kc & k(-a) \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & -ka \end{bmatrix}$.

The sum of the diagonals is

$ka + (-ka) = ka - ka = 0$.

kA is in T which is closed under

Scalar multiplication.

T is a subspace of $M_{2\times 2}$.

$$T = \left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \;\middle|\; a, b, c \in R \right\}$$

Show that the set $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$ is a basis for T.

For $A = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ in T

$$A = a \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

So $\mathcal{B}$ spans T. To show that $\mathcal{B}$ is linearly independent, consider

$$x_1 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + x_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 & x_2 \\ x_3 & -x_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Comparing entries $x_1 = 0$, $x_2 = 0$, $x_3 = 0$

$-x_1 = 0$.

This trivial solution is the only solution. So B is lin. independent.

Hence B is a basis for T.

It can be proved that if a subspace S of a vector space V has a basis with k vectors in it ($1 \leq k < \infty$) then every basis of S has exactly k vectors in it.

Dimension

Let S be a subspace of a vector space V. We define the **dimension** of S as follows:

- If $S = \{\vec{0}_V\}$, then we define $\text{dim}(S) = 0$.
- If S has a basis consisting of k vectors, where $k < \infty$, then we define $\text{dim}(S) = k$.
- If S is not spanned by any finite set of vectors, then we say that S is infinite dimensional.

The spaces R^∞ and $F(D)$ are examples of infinite dimensional vectors spaces. Bases for such vector spaces are known to exist but are not particularly useful. We can readily construct finite dimensional subspaces of infinite dimensional vector spaces.

$$T = \left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \;\middle|\; a, b, c \in R \right\}$$

We saw that $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$ is a basis for T. It follows that

$$\dim(T) = \underline{\quad 3 \quad}$$

3 basis elements

Example $M_{2\times 2}$

The set $\mathcal{E} = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ is a basis for $M_{2\times 2}$

$$E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

(Proof is left as an exercise.)

It follows that $\dim(M_{2\times 2}) =$ ___4___.

four basis elements

Coordinate Vectors

Suppose that V is a vector space and suppose that S is a finite dimensional subspace of V. Suppose that $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_k\}$ is an ordered basis for S. Then the (unique) vector

$$[\vec{x}]_{\mathcal{B}} = \langle c_1, c_2, \dots, c_k \rangle \in R^k$$

such that

$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k$$

is called the **coordinate vector** of $\vec{x}$ with respect to the ordered basis $\mathcal{B}$.

Remark 1: Note that regardless of what sorts of objects the vectors in S are, the coordinate vectors are vectors in R^k. They're real k-tuples!

Remark 2: We'll talk about using coordinate vectors more in Section 4.8. Now, we go back to examples from Section 4.7.

Example

Consider the subspace T of $M_{2\times 2}$ of trace-free matrices, and consider the ordered basis $\mathcal{B}_T$ for T, where $\mathcal{B}_T = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$.

1. Find $[A]_{\mathcal{B}_T}$ if $A = \begin{bmatrix} -3 & 2 \\ 4 & 3 \end{bmatrix}$. $= c_1 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

$[A]_{\mathcal{B}_T} = \langle c_1, c_2, c_3 \rangle$ $\qquad [A]_{\mathcal{B}_T} = \langle -3, 2, 4 \rangle$

2. Find $[B]_{\mathcal{B}_T}$ if $B = \begin{bmatrix} 5 & 0 \\ -1 & -5 \end{bmatrix}$. $[B]_{\mathcal{B}_T} = \langle 5, 0, -1 \rangle$

3. Find a vector C in T if $[C]_{\mathcal{B}_T} = \langle 1, 7, 6 \rangle$,

$$C = 1 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + 7 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 6 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 7 \\ 6 & -1 \end{bmatrix}$$

$$\mathcal{B}_T = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}, \quad A = \begin{bmatrix} -3 & 2 \\ 4 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 0 \\ -1 & -5 \end{bmatrix}$$

$$[A]_{\mathcal{B}_T} = \langle -3, 2, 4 \rangle, \quad [B]_{\mathcal{B}_T} = \langle 5, 0, -1 \rangle$$

1. Show that $[A+B]_{\mathcal{B}_T} = [A]_{\mathcal{B}_T} + [B]_{\mathcal{B}_T}$.

$$A+B = \begin{bmatrix} -3 & 2 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ -1 & -5 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 3 & -2 \end{bmatrix} \qquad [A+B]_{\mathcal{B}_T} = \langle 2, 2, 3 \rangle$$

$$[A]_{\mathcal{B}_T} + [B]_{\mathcal{B}_T} = \langle -3, 2, 4 \rangle + \langle 5, 0, -1 \rangle = \langle 2, 2, 3 \rangle$$

equal

2. Show that $[2A]_{\mathcal{B}_T} = 2\,[A]_{\mathcal{B}_T}$.

$$2A = \begin{bmatrix} -6 & 4 \\ 8 & 6 \end{bmatrix} \qquad [2A]_{\mathcal{B}_T} = \langle -6, 4, 8 \rangle$$

equal

$$2\,[A]_{\mathcal{B}_T} = 2\langle -3, 2, 4 \rangle = \langle -6, 4, 8 \rangle$$

The following lemmas appear in section 4.8

Lemma

Suppose that S is a subspace of a vector space V and $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_k\}$ is an ordered basis of S. If $\vec{x}$ and $\vec{y}$ are any two vectors in S and c is any scalar then

1. $[\vec{x} + \vec{y}]_{\mathcal{B}} = [\vec{x}]_{\mathcal{B}} + [\vec{y}]_{\mathcal{B}}$ and
2. $[c\vec{x}]_{\mathcal{B}} = c\,[\vec{x}]_{\mathcal{B}}$.

Lemma

Suppose S is a subspace of a vector space V and $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_k\}$ is an ordered basis of S. Then $\vec{0}_V$ is the only vector in S that has coordinate vector $\vec{0}_k$. In other words, the following statement holds for all vectors $\vec{x} \in S$:

$$[\vec{x}]_{\mathcal{B}} = \vec{0}_k \text{ if and only if } \vec{x} = \vec{0}_V.$$

Section 4.7.4.2 Function Spaces $C^n(I)$

Infinite Dimensional Subspaces of $F(I)$

Let I be an interval in R. The set

$$C^n(I)$$

is the set of all functions $f : I \to R$ that are n times continuously differentiable on I.

- $C^0(I)$ contains functions that are continuous on I.
- $C^1(I)$ contains functions that have a continuous derivative on I.
- $C^2(I)$ contains functions that have a continuous first and second derivatives on I...

The set $C^\infty(I)$ contains functions with continuous derivatives of all orders on I. Some common examples from $C^\infty(R)$ include

$$\tan^{-1}(x), \quad \sin(x), \quad e^x$$

all polynomials, etc .

Example

Consider the set $\mathcal{B}_2 = \{1, x, x^2\}$ in $C^\infty(R)$. Show that $\mathcal{B}_2$ is a basis for $\text{Span}(\mathcal{B}_2)$.

$f_0(x) = 1, \quad f_1(x) = x, \quad f_2(x) = x^2$

Since were interested in Span (B_2), we only have to show that B_2 is linearly independent. Consider

$$c_0(1) + c_1 x + c_2 x^2 = z(x) \qquad *$$

where $z(x) = 0$ for all real x.

$*$ must hold for all real x. So it holds at $x = 0$. .

$\mathcal{B}_2 = \{1, x, x^2\}^2$

For $x=0$, $\ast$ becomes

$$c_0(1) + c_1(0) + c_2(0^2) = z(0) = 0$$

$\Rightarrow c_0 = 0$. $\ast$ holds when $x=1$ and when $x=-1$.

$\ast$ becomes
$$\begin{aligned} c_1(1) + c_2(1^2) &= z(1) = 0 \\ c_1(-1) + c_2((-1)^2) &= z(-1) = 0 \end{aligned} \Rightarrow$$

$$\begin{aligned} c_1 + c_2 &= 0 \\ -c_1 + c_2 &= 0 \end{aligned} \Rightarrow c_1 = 0 \text{ and } c_2 = 0.$$

$\ast$ has only the trivial solution. So $\mathcal{B}_2 = \{1, x, x^2\}$ is linearly independent in $C^\infty(\mathbb{R})$ and hence is a basis of $\text{Span}(\mathcal{B}_2)$.

[2] It can be shown that for $n \geq 1$ the set $\{1, x, x^2, \ldots, x^n\}$ is linearly independent in $C^\infty(R)$.