November 6 Math 2306 sec. 51 Fall 2024

Section 14: Inverse Laplace Transforms

We're going to use the Laplace transform to solve IVPs. So in addition to taking a transform to go from a function of t to a function of s, we'll want to go backwards.

Question: Given $F(s)$ can we find a function $f(t)$ such that $\mathscr{L}\{f(t)\} = F(s)$?

Inverse Laplace Transform

Let $F(s)$ be a function. An **inverse Laplace transform** of F is a piecewise continuous function $f(t)$ provided $\mathscr{L}\{f(t)\} = F(s)$. We will use the notation

$$\mathscr{L}^{-1}\{F(s)\} = f(t) \quad \text{if} \quad \mathscr{L}\{f(t)\} = F(s).$$

A Table of Inverse Laplace Transforms

- $\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1$
- $\mathscr{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$, for $n = 1, 2, \ldots$
- $\mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$
- $\mathscr{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos kt$
- $\mathscr{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin kt$

The inverse Laplace transform is also linear so that

$$\mathscr{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha f(t) + \beta g(t)$$

Using a Table

When using a table of Laplace transforms, the expression must match exactly. For example,

$$\mathscr{L}\left\{t^n\right\} = \frac{n!}{s^{n+1}}$$

so

$$\mathscr{L}^{-1}\left\{\frac{3!}{s^4}\right\} = t^3.$$

Note that $n = 3$, so there must be 3! in the numerator and the power $4 = 3 + 1$ on s.

Remark: The function $F(s)$ often requires some amount of manimpulation to get it to look like a table entry. There are a few common tricks of the trade to taking inverse Laplace transforms.

Find the Inverse Laplace Transform

$$\mathscr{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n,$$

(a) $\mathscr{L}^{-1}\left\{\dfrac{1}{s^7}\right\}$ If $n+1=7$, then $n=6$.

Note: $\dfrac{1}{s^7} = \dfrac{6!}{6!}\dfrac{1}{s^7} = \dfrac{1}{6!}\dfrac{6!}{s^7}$

$$\mathscr{L}^{-1}\left\{\frac{1}{s^7}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{6!}\frac{6!}{s^7}\right\} = \frac{1}{6!}\mathscr{L}^{-1}\left\{\frac{6!}{s^7}\right\}$$

$$= \frac{1}{6!}t^6$$

Example: Evaluate

(b) $\mathscr{L}^{-1}\left\{\frac{s+1}{s^2+9}\right\}$

$\mathscr{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos kt$

$\mathscr{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin kt$

$= \mathscr{L}^{-1}\left\{\frac{s}{s^2+9} + \frac{1}{s^2+9}\right\}$

$= \mathscr{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{s^2+3^2}\right\}$

$= \mathscr{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} + \frac{1}{3}\mathscr{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\}$

$= \cos 3t + \frac{1}{3}\sin 3t$

Example: Evaluate

(c) $\mathscr{L}^{-1}\left\{\dfrac{s-8}{s^2-2s}\right\}$

Consider evaluating

$$\int \frac{x-8}{x^2-2x}\,dx$$

we need a partial fraction decomp.

$$\frac{s-8}{s^2-2s} = \frac{s-8}{s(s-2)} = \frac{A}{s} + \frac{B}{s-2}$$

clear fractions

$$s-8 = A(s-2) + Bs$$

Set $s=0$ $\quad -8 = A(-2) \Rightarrow A = 4$

$s=2$ $\quad 2-8 = -6 = 2B \Rightarrow B = -3$

$$\mathscr{L}^{-1}\left\{\frac{s-8}{s^2-2s}\right\} = \mathscr{L}^{-1}\left\{\frac{4}{s} - \frac{3}{s-2}\right\}$$

$$= 4\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} - 3\mathscr{L}^{-1}\left\{\frac{1}{s-2}\right\}$$

$$= 4(1) - 3e^{2t}$$

$$= 4 - 3e^{2t}$$

$$\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1$$

$\mathscr{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$, for

$$\mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$$

Convolutions & Laplace Transforms

Question: Consider $\mathscr{L}^{-1}\left\{\dfrac{1}{s^2+8s+15}\right\}$. Is it useful to note that

$$\frac{1}{s^2+8s+15} = \left(\frac{1}{s+3}\right)\left(\frac{1}{s+5}\right)?$$

As an integral, it is clear that the transform or inverse transform of a product is **NOT** the product of the transforms. That is

$$\mathscr{L}\{f(t)g(t)\} \neq \mathscr{L}\{f(t)\}\mathscr{L}\{g(t)\}$$

and similarly

$$\mathscr{L}^{-1}\{F(s)G(s)\} \neq \mathscr{L}^{-1}\{F(s)\}\mathscr{L}^{-1}\{G(s)\}$$

There is a special type of *product* of functions that can be used to evaluate an inverse transform of the form $\mathscr{L}^{-1}\{F(s)G(s)\}$. The special product is called a **convolution**

Convolution

Definition

Let f and g be piecewise continuous on $[0, \infty)$ and of exponential order c for some $c \geq 0$. The **convolution** of f and g is denoted by $f * g$ and is defined by

$$(f * g)(t) = \int_0^t f(\tau) g(t - \tau)\, d\tau$$

Remark: In a more general setting in which functions of interest are defined on $(-\infty, \infty)$, the convolution is typically defined as

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau) g(t - \tau)\, d\tau$$

If the functions $f(t)$ and $g(t)$ are assigned to take the value of zero for $t < 0$, this definition reduces to the one given here.

Example

Compute the convolution of $f(t) = e^{-3t}$ and $g(t) = e^{-5t}$.

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$$

$f(t) = e^{-3t}$, $f(\tau) = e^{-3\tau}$

$g(t) = e^{-5t}$, $g(t-\tau) = e^{-5(t-\tau)}$

$$(f * g)(t) = \int_0^t e^{-3\tau} e^{-5(t-\tau)} \, d\tau$$

$$= \int_0^t e^{-3\tau} e^{-5t} \cdot e^{5\tau} \, d\tau$$

$$= e^{-5t} \int_0^t e^{-3\tau} e^{5\tau} \, d\tau$$

$$= e^{-5t} \int_0^t e^{2\tau} \, d\tau$$

$$= e^{-5t} \left[\frac{1}{2} e^{2\tau} \right|_0^t$$

$$= e^{t} \left[\frac{1}{2} e^{2t} - \frac{1}{2} e^{2(0)} \right]$$

$$= \frac{1}{2} e^{-3t} - \frac{1}{2} e^{-5t}$$

$$\left(e^{-3t} * e^{-5t}\right)(t) = \frac{1}{2} e^{-3t} - \frac{1}{2} e^{-5t}$$

Laplace Transforms & Convolutions

The Laplace transform of a convolution is related to the product of Laplace transforms.

Theorem

Suppose $\mathscr{L}\{f(t)\} = F(s)$ and $\mathscr{L}\{g(t)\} = G(s)$. Then

$$\mathscr{L}\{f * g\} = F(s)G(s)$$

Theorem

Suppose $\mathscr{L}^{-1}\{F(s)\} = f(t)$ and $\mathscr{L}^{-1}\{G(s)\} = g(t)$. Then

$$\mathscr{L}^{-1}\{F(s)G(s)\} = (f * g)(t)$$

Remark: This is the same theorem stated first from the perspective of a Laplace transform and then from the perspective of an inverse Laplace transform.

Example

Use the convolution to evaluate

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2+8s+15}\right\} = \mathscr{L}^{-1}\left\{\left(\frac{1}{s+3}\right)\left(\frac{1}{s+5}\right)\right\}$$

Let $F(s) = \dfrac{1}{s+3}$ and $G(s) = \dfrac{1}{s+5}$

Set $f(t) = \mathscr{L}^{-1}\{F(s)\} = \mathscr{L}^{-1}\left\{\dfrac{1}{s+3}\right\} = e^{-3t}$

$g(t) = \mathscr{L}^{-1}\{G(s)\} = \mathscr{L}^{-1}\left\{\dfrac{1}{s+5}\right\} = e^{-5t}$

$\mathscr{L}^{-1}\{F(s)G(s)\} = (f * g)(t)$

$$\boxed{\mathcal{L}^{-1}\left\{\frac{1}{s^2+8s+15}\right\} = \frac{1}{2}e^{-3t} - \frac{1}{2}e^{-5t}}$$

$$\frac{1}{s^2+8s+15} = \frac{A}{s+3} + \frac{B}{s+5}$$

$$1 = A(s+5) + B(s+3)$$

$s = -3 \qquad 1 = 2A \quad \Rightarrow \quad A = \frac{1}{2}$

$s = -5 \qquad 1 = -2B \quad \Rightarrow \quad B = \frac{-1}{2}$

Example

Evaluate $\mathscr{L}\left\{\int_0^t \tau^6 e^{-4(t-\tau)}\,d\tau\right\}$ $= \frac{6!}{s^7}\left(\frac{1}{s+4}\right) = \frac{6!}{s^7(s+4)}$

$f(t) = t^6$, $g(t) = e^{-4t}$

$(f*g)(t) = \int_0^t \tau^6 e^{-4(t-\tau)}\,d\tau$

$\mathcal{L}\{f*g\} = F(s)\,G(s)$

$$F(s) = \frac{6!}{s^7} \qquad G(s) = \frac{1}{s+4}$$

$$\mathcal{L}\{t^6\} = \frac{6!}{s^{6+1}} \qquad \mathcal{L}\{e^{-4t}\} = \frac{1}{s-(-4)}$$