November 6 Math 2306 sec. 51 Spring 2023

Section 14: Inverse Laplace Transforms

Definition:

Let $f(t)$ be piecewise continuous on $[0,\infty)$. The Laplace transform of f, denoted $\mathscr{L}\{f(t)\}$ or $F(s)$ is given by.

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt.$$

Inverse Laplace Transform

Let $F(s)$ be a function. An **inverse Laplace transform** of F is a piecewise continuous function $f(t)$ provided $\mathscr{L}\{f(t)\} = F(s)$. We will use the notation

$$\mathscr{L}^{-1}\{F(s)\} = f(t) \quad \text{if} \quad \mathscr{L}\{f(t)\} = F(s).$$

Example: Use a Table to Evaluate

(c) $\mathscr{L}^{-1}\left\{\dfrac{s+4}{s^2-4}\right\}$

Start w/ partial fractions.

$$\frac{s+4}{(s-2)(s+2)} = \frac{A}{s-2} + \frac{B}{s+2}$$ Clear fractions

$$s+4 = A(s+2) + B(s-2)$$

Set $s=2$ $\quad 2+4 = A(2+2) + B(2-2)$

$$6 = 4A \implies A = \frac{6}{4} = \frac{3}{2}$$

$s=-2$ $\quad -2+4 = A(-2+2) + B(-2-2)$

$$2 = -4B \implies B = \frac{2}{-4} = -\frac{1}{2}$$

$$\frac{s+4}{s^2-4} = \frac{\frac{3}{2}}{s-2} - \frac{\frac{1}{2}}{s+2}$$

$$\mathcal{L}^{-1}\left\{\frac{s+4}{s^2-4}\right\} = \mathcal{L}^{-1}\left\{\frac{\frac{3}{2}}{s-2} - \frac{\frac{1}{2}}{s+2}\right\}$$

$$= \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-(-2)}\right\}$$

$$= \frac{3}{2}e^{2t} - \frac{1}{2}e^{-2t}$$

A Look Ahead: Solving IVPs

If $f(t)$ is defined on $[0, \infty)$, is differentiable, and has Laplace transform $F(s) = \mathscr{L}\{f(t)\}$, then*

$$\mathscr{L}\{f'(t)\} = sF(s) - f(0)$$

Use this result to solve the initial value problem

$$y'(t) + 4y(t) = 16t, \quad y(0) = 1$$

We'll assume that the solution $y(t)$ has a Laplace transform. Let $Y(s) = \mathscr{L}\{y(t)\}$.

$$y'(t) + 4y(t) = 16t, \quad y(0) = 1$$

Take the transform of both sides

$\mathcal{L}\{y' + 4y\} = \mathcal{L}\{16t\}$

$\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = 16\mathcal{L}\{t\}$

use $Y(s) = \mathcal{L}\{y\}$ and $sY(s) - y(0) = \mathcal{L}\{y'\}$

$sY(s) - y(0) + 4Y(s) = 16\left(\frac{1!}{s^2}\right) = \frac{16}{s^2}$

Isolate $Y(s)$

sub in $y(0) = 1$

$sY(s) - 1 + 4Y(s) = \frac{16}{s^2}$

$$(s+4)Y(s) = \frac{16}{s^2} + 1 = \frac{16+s^2}{s^2}$$

$$\Rightarrow \quad Y(s) = \frac{16+s^2}{s^2(s+4)}$$

$\mathcal{L}\{y(t)\} = Y(s)$, so the solution to the IVP

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}.$$

Partial fractions

$(s^2(s+4))$ $$\frac{16+s^2}{s^2(s+4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+4}$$ clear fractions

$$16+s^2 = As(s+4) + B(s+4) + Cs^2$$

$$= A(s^2+4s) + B(s+4) + Cs^2$$

$$\underline{s^2} + \underline{0s} + \underline{16} = \underline{(A+C)s^2} + \underline{(4A+B)s} + \underline{4B}$$

$A + C = 1$ $\quad 4A + B = 0$ $\quad 4B = 16$

$C = 1 - A = 2$ $\quad A = -\frac{1}{4}B = -1$ $\quad B = 4$

$$Y(s) = \frac{-1}{s} + \frac{4}{s^2} + \frac{2}{s+4}$$

The solution to the IVP

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = -\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + 4\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\}$$

$$\boxed{y(t) = -1 + 4t + 2e^{-4t}}$$