November 8 Math 2306 sec. 51 Fall 2021

Section 16: Laplace Transforms of Derivatives and IVPs

We can solve a system of ODEs using Laplace transforms. Here, we'll consider systems that are

- linear,
- having initial conditions at $t = 0$, and
- constant coefficient.

Let's see it in action (i.e. with a couple of examples).

Example

Solve the system of equations

$$\begin{aligned}\frac{dx}{dt} &= -2x - 2y + 60, \quad x(0) = 0 \\ \frac{dy}{dt} &= -2x - 5y + 60, \quad y(0) = 0\end{aligned}$$

Let $X(s) = \mathcal{L}\{x(t)\}$ and $Y(s) = \mathcal{L}\{y(t)\}$.

Take transform of both ODE

$$\mathcal{L}\{x'\} = \mathcal{L}\{-2x - 2y + 60\}$$

$$\mathcal{L}\{y'\} = \mathcal{L}\{-2x - 5y + 60\}$$

$$sX(s) - \cancel{x(0)}^{\,0} = -2X(s) - 2Y(s) + \frac{60}{s}$$

$$sY(s) - \cancel{y(0)}^{\,0} = -2X(s) - 5Y(s) + \frac{60}{s}$$

Example Continued...[1]

pull all X, Y terms to the left

$$sX + 2X + 2Y = \frac{60}{s}$$

$$sY + 2X + 5Y = \frac{60}{s}$$

$$(s+2)X + 2Y = \frac{60}{s}$$

$$2X + (s+5)Y = \frac{60}{s}$$

using Crammer's rule

$$A = \begin{bmatrix} s+2 & 2 \\ 2 & s+5 \end{bmatrix}, \quad A_X = \begin{bmatrix} \frac{60}{s} & 2 \\ \frac{60}{s} & s+5 \end{bmatrix}, \quad A_Y = \begin{bmatrix} s+2 & \frac{60}{s} \\ 2 & \frac{60}{s} \end{bmatrix}$$

[1] Crammer's Rule will be helpful at some point.

$$\det(A) = (s+2)(s+5) - 4 = s^2 + 7s + 10 - 4 = s^2 + 7s + 6$$
$$= (s+6)(s+1)$$

$$\det(A_x) = \frac{60}{s}(s+5) - \frac{60}{s}\cdot 2 = \frac{60}{s}(s+5-2) = \frac{60}{s}(s+3)$$

$$\det(A_y) = (s+2)\frac{60}{s} - 2\,\frac{60}{s} = \frac{60}{s}(s+2-2) = \frac{60}{s}s = 60$$

$$X(s) = \frac{\frac{60}{s}(s+3)}{(s+6)(s+1)} = \frac{60(s+3)}{s(s+6)(s+1)}$$

and

$$Y(s) = \frac{60}{(s+6)(s+1)}$$

Partial fraction decomps are required to take the inverse transforms.

$$\frac{60(s+3)}{s(s+6)(s+1)} = \frac{A}{s} + \frac{B}{s+6} + \frac{C}{s+1}$$

$$60(s+3) = A(s+6)(s+1) + Bs(s+1) + Cs(s+6)$$

set $s=0$ $\quad 60(3) = A(6)(1) \Rightarrow A = \frac{60(3)}{6} = 30$

$s=-6$ $\quad 60(-3) = B(-6)(-5) \Rightarrow B = \frac{60(-3)}{+30} = -6$

$s=-1$ $\quad 60(2) = C(-1)(5) \Rightarrow C = \frac{60(2)}{-5} = -24$

so $\quad X(s) = \frac{30}{s} - \frac{6}{s+6} - \frac{24}{s+1}$

Y is done similarly

$$Y(s) = \frac{12}{s+1} - \frac{12}{s+6}$$

The solution to the IVP

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = 30 - 6e^{-6t} - 24e^{-t}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = 12e^{-t} - 12e^{-6t}$$

$$\boxed{\begin{aligned} x(t) &= 30 - 6e^{-6t} - 24e^{-t} \\ y(t) &= 12e^{-t} - 12e^{-6t} \end{aligned}}$$

Example

Use the Laplace transform to solve the system of equations

$$\begin{aligned} x''(t) &= y, \quad x(0)=1, \quad x'(0)=0 \\ y'(t) &= x, \quad y(0)=1 \end{aligned}$$

Let $X(s) = \mathcal{L}\{x(t)\}$ and $Y(s) = \mathcal{L}\{y(t)\}$

$\mathcal{L}\{x''\} = \mathcal{L}\{y\}$

$\mathcal{L}\{y'\} = \mathcal{L}\{x\}$

$s^2 X - s\,\cancel{x(0)}^{\,1} - \cancel{x'(0)}^{\,0} = Y(s)$

$sY(s) - \cancel{y(0)}^{\,1} = X(s)$

$$s^2 X - s = Y \qquad\qquad s^2 X - Y = s$$
$$sY - 1 = X \quad \Longrightarrow \quad -X + sY = 1$$

Again using Crammer's rule

$$A = \begin{bmatrix} s^2 & -1 \\ -1 & s \end{bmatrix}, \quad A_X = \begin{bmatrix} s & -1 \\ 1 & s \end{bmatrix}, \quad A_Y = \begin{bmatrix} s^2 & s \\ -1 & 1 \end{bmatrix}$$

$$\det(A) = s^3 - 1, \quad \det(A_X) = s^2 + 1, \quad \det(A_Y) = s^2 + s$$

Hence $X(s) = \dfrac{s^2+1}{s^3-1}$, $Y(s) = \dfrac{s^2+s}{s^3-1}$.

Note $s^3 - 1 = (s-1)(s^2+s+1)$, s^2+s+1 is irreducible

$$X(s) = \frac{s^2+1}{(s-1)(s^2+s+1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+s+1}.$$

$$Y(s) = \frac{s^2+s}{(s-1)(s^2+s+1)} = \frac{D}{s-1} + \frac{Es+F}{s^2+s+1}$$

after some work

$$X = \frac{\frac{2}{3}}{s-1} + \frac{\frac{1}{3}(s-1)}{s^2+s+1}$$

$$Y = \frac{2/3}{s-1} + \frac{\frac{1}{3}(s+2)}{s^2+s+1}$$

Completing the square on s^2+s+1

$$s^2+s+\frac{1}{4}-\frac{1}{4}+1 = \left(s+\frac{1}{2}\right)^2+\frac{3}{4} = \left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2$$

$$X = \frac{\frac{2}{3}}{s-1} + \frac{\frac{1}{3}(s-1)}{s^2+s+1}$$

$$\varphi = \frac{2/3}{s-1} + \frac{\frac{1}{3}(s+2)}{s^2+s+1}$$

we need $s+\frac{1}{2}$ everywhere that s is

$$s-1 = s+\frac{1}{2}-\frac{3}{2}$$

$$s+2 = s+\frac{1}{2}+\frac{3}{2}$$

$$X = \frac{2/3}{s-1} + \frac{1}{3}\,\frac{s+\frac{1}{2}}{(s+1/2)^2+\left(\frac{\sqrt{3}}{2}\right)^2} - \frac{1}{3}\,\frac{3}{2}\,\frac{1}{(s+\frac{1}{2})^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$Y = \frac{2/3}{s-1} + \frac{1}{3}\,\frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+\left(\frac{\sqrt{3}}{2}\right)^2} + \frac{1}{3}\left(\frac{3}{2}\right)\cdot\frac{1}{(s+\frac{1}{2})^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

we'll finish next time

Crammer's Rule

Crammer's Rule is a method to solve an algebraic linear system of the form

$$\begin{array}{ccccc} ax & + & by & = & e \\ cx & + & dy & = & f \end{array}$$

Define the following matrices

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad A_x = \begin{bmatrix} e & b \\ f & d \end{bmatrix}, \quad \text{and} \quad A_y = \begin{bmatrix} a & e \\ c & f \end{bmatrix}.$$

If $\det(A) \neq 0$, then the system is uniquely solvable and the solution

$$x = \frac{\det(A_x)}{\det(A)} \quad \text{and} \quad y = \frac{\det(A_y)}{\det(A)}.$$

Crammer's Rule 3×3 case

$$\begin{array}{ccccccc} a_{11}x & + & a_{12}y & + & a_{13}z & = & b_1 \\ a_{21}x & + & a_{22}y & + & a_{23}z & = & b_2 \\ a_{31}x & + & a_{32}y & + & a_{33}z & = & b_3 \end{array}$$

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \qquad A_x = \begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix}$$

$$A_y = \begin{bmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{bmatrix}, \qquad A_z = \begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{bmatrix}$$

If $\det(A) \neq 0$, then the solution to the system

$$x = \frac{\det(A_x)}{\det(A)}, \quad y = \frac{\det(A_y)}{\det(A)}, \quad \text{and} \quad z = \frac{\det(A_z)}{\det(A)}.$$