November 8 Math 2306 sec. 51 Spring 2023

Section 14: Convolutions

As an integral, it is clear that the transform or inverse transform of a product is **NOT** the product of the transforms. That is

$$\mathscr{L}\{f(t)g(t)\} \neq \mathscr{L}\{f(t)\}\mathscr{L}\{g(t)\}$$

and similarly

$$\mathscr{L}^{-1}\{F(s)G(s)\} \neq \mathscr{L}^{-1}\{F(s)\}\mathscr{L}^{-1}\{G(s)\}$$

There is a special type of *product* of functions that can be used to evaluate an inverse transform of the form $\mathscr{L}^{-1}\{F(s)G(s)\}$. The special product is called a **convolution**

Convolution

Definition

Let f and g be piecewise continuous on $[0, \infty)$ and of exponential order c for some $c \geq 0$. The **convolution** of f and g is denoted by $f * g$ and is defined by

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$$

Remark: In a more general setting in which functions of interest are defined on $(-\infty, \infty)$, the convolution is typically defined as

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)\,d\tau$$

If the functions $f(t)$ and $g(t)$ are assigned to take the value of zero for $t < 0$, this definition reduces to the one given here.

Example

Compute the convolution of $f(t) = e^{-3t}$ and $g(t) = e^{-5t}$.

By definition

$$(f * g)(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

$$f(t) = e^{-3t}, \quad f(\tau) = e^{-3\tau}$$

$$g(t) = e^{-5t}, \quad g(t-\tau) = e^{-5(t-\tau)} = e^{-5t} \cdot e^{5\tau}$$

$$(f * g)(t) = \int_0^t e^{-3\tau}\, e^{-5t} \cdot e^{5\tau}\, d\tau$$

$$= e^{-5t} \int_0^t e^{2\tau} \, d\tau$$

$$= e^{-5t} \left[\frac{1}{2} e^{2\tau} \right|_0^t$$

$$= e^{-5t} \left[\frac{1}{2} e^{2t} - \frac{1}{2} e^{0} \right]$$

$$(f * g)(t) = \frac{1}{2} e^{-3t} - \frac{1}{2} e^{-5t}$$

for $f(t) = e^{-3t}$, $g(t) = e^{-5t}$

Laplace Transforms & Convolutions

The Laplace transform of a convolution is related to the product of Laplace transforms.

Theorem

Suppose $\mathscr{L}\{f(t)\} = F(s)$ and $\mathscr{L}\{g(t)\} = G(s)$. Then

$$\mathscr{L}\{f * g\} = F(s)G(s)$$

Theorem

Suppose $\mathscr{L}^{-1}\{F(s)\} = f(t)$ and $\mathscr{L}^{-1}\{G(s)\} = g(t)$. Then

$$\mathscr{L}^{-1}\{F(s)G(s)\} = (f * g)(t)$$

Remark: This is the same theorem stated first from the perspective of a Laplace transform and then from the perspective of an inverse Laplace transform.

Example

Use the convolution to evaluate

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2+8s+15}\right\} = \mathscr{L}^{-1}\left\{\left(\frac{1}{s+3}\right)\left(\frac{1}{s+5}\right)\right\}$$

Let $F(s) = \frac{1}{s+3}$ and $G(s) = \frac{1}{s+5}$

so $\frac{1}{s^2+8s+15} = F(s)G(s)$

Find $\mathscr{L}^{-1}\{F(s)\}$ and $\mathscr{L}^{-1}\{G(s)\}$

$f(t) = \mathscr{L}^{-1}\left\{\frac{1}{s+3}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{s-(-3)}\right\} = e^{-3t}$

$$g(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s-(-5)}\right\} = e^{-5t}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+8s+15}\right\} = (f * g)(t)$$

where $f(t) = e^{-3t}$ and $g(t) = e^{-5t}$

$$= \frac{1}{2}e^{-3t} - \frac{1}{2}e^{-5t}$$

Example

$\mathcal{L}\{f*g\} = F(s)G(s)$

Evaluate $\mathscr{L}\left\{\int_0^t \tau^6 e^{-4(t-\tau)}\,d\tau\right\}$

If $f(\tau) = \tau^6$ then $f(t) = t^6$

If $g(t-\tau) = e^{-4(t-\tau)}$, $g(t) = e^{-4t}$

$\mathcal{L}\{t^6\} = \frac{6!}{s^7}$, $\mathcal{L}\{e^{-4t}\} = \frac{1}{s-(-4)} = \frac{1}{s+4}$

$\mathcal{L}\left\{\int_0^t \tau^6 e^{-4(t-\tau)}\,d\tau\right\} = \frac{6!}{s^7}\cdot\left(\frac{1}{s+4}\right) = \frac{6!}{s^7(s+4)}$

Example

Evaluate the inverse Laplace transform in two ways, using a partial fraction decomposition and using a convolution.

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2(s+1)}\right\}$$

To save time, here is a decomposition of the argument.

$$\frac{1}{s^2(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1}.$$

After some algebra, we find that $A = -1$, $B = 1$ and $C = 1$.

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{-1}{s} + \frac{1}{s^2} + \frac{1}{s+1}\right\}$$

$$= \underline{-1} + t + e^{-t}$$

$$\frac{1}{s^2(s+1)} = F(s)\,G(s) \quad \text{if} \quad F(s) = \frac{1}{s^2},\ G(s) = \frac{1}{s+1}$$

$$\mathcal{L}^{-1}\{F(s)\,G(s)\} = (f*g)(t)$$

where $f(t) = \mathcal{L}^{-1}\{F(s)\} = t$ and

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = e^{-t}$$

$$(f*g)(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

$$= \int_0^t \tau \, e^{-(t-\tau)} \, d\tau$$

$$= \int_0^t \tau \, e^{-t} \cdot e^{\tau} \, d\tau$$

$$= e^{-t} \int_0^t \tau e^{\tau} \, dt$$

$u = \tau \quad du = d\tau$

$v = e^{\tau} \quad dv = e^{\tau} d\tau$

$$= e^{-t} \left[\tau e^{\tau} \Big|_0^t - \int_0^t e^{\tau} \, d\tau \right]$$

$$= e^{-t} \left[\tau e^{\tau} - e^{\tau} \Big|_0^t \right.$$

$$= e^{-t} \left[t e^{t} - e^{t} - \left(0 e^{0} - e^{0} \right) \right]$$

$$= e^{-t}\left(te^{t} - e^{t} + 1\right)$$

$$= t e^{-t} e^{t} - e^{-t} e^{t} + e^{-t}$$

$$= t - 1 + e^{-t}$$