November 8 Math 2306 sec. 52 Fall 2021

Section 16: Laplace Transforms of Derivatives and IVPs

We can solve a system of ODEs using Laplace transforms. Here, we'll consider systems that are

- linear,
- having initial conditions at $t = 0$, and
- constant coefficient.

Let's see it in action (i.e. with a couple of examples).

Example

Solve the system of equations

$$\begin{aligned}\frac{dx}{dt} &= -2x - 2y + 60, \quad x(0) = 0\\ \frac{dy}{dt} &= -2x - 5y + 60, \quad y(0) = 0\end{aligned}$$

Let $X(s) = \mathscr{L}\{x(t)\}$ and $Y(s) = \mathscr{L}\{y(t)\}$

$$\mathscr{L}\{x'\} = \mathscr{L}\{-2x - 2y + 60\}$$

$$\mathscr{L}\{y'\} = \mathscr{L}\{-2x - 5y + 60\}$$

Example Continued...[1]

$$sX(s) - x(0) = -2X(s) - 2Y(s) + \frac{60}{s}$$

$$sY(s) - y(0) = -2X(s) - 5Y(s) + \frac{60}{s}$$

Use $x(0)=0$, and $y(0)=0$

$$sX(s) + 2X(s) + 2Y(s) = \frac{60}{s}$$

$$sY(s) + 2X(s) + 5Y(s) = \frac{60}{s}$$

$$(s+2)X(s) + 2Y(s) = \frac{60}{s}$$

$$2X(s) + (s+5)Y(s) = \frac{60}{s}$$

[1] Crammer's Rule will be helpful at some point.

Using Crammer's rule

$$A = \begin{bmatrix} s+2 & 2 \\ 2 & s+5 \end{bmatrix} \quad A_x = \begin{bmatrix} \frac{60}{s} & 2 \\ \frac{60}{s} & s+5 \end{bmatrix}, \quad A_y = \begin{bmatrix} s+2 & \frac{60}{s} \\ 2 & \frac{60}{s} \end{bmatrix}$$

$$\det(A) = (s+2)(s+5) - 4 = s^2 + 7s + 10 - 4 = s^2 + 7s + 6$$

$$= (s+1)(s+6)$$

$$\det(A_x) = \frac{60}{s}(s+5) - \frac{60}{s}(2) = \frac{60}{s}(s+5-2) = \frac{60}{s}(s+3)$$

$$\det(A_y) = (s+2)\frac{60}{s} - 2\frac{60}{s} = \frac{60}{s}(s+2-2) = \frac{60}{s}s = 60$$

$$X = \frac{\det(A_x)}{\det(A)} = \frac{\frac{60}{s}(s+3)}{(s+1)(s+6)} = \frac{60(s+3)}{s(s+1)(s+6)}$$

$$Y = \frac{\det(A_Y)}{\det(A)} = \frac{60}{(s+1)(s+6)}$$

We have to do partial fraction decomps to take the inverse transform.

$$X(s) = \frac{60(s+3)}{s(s+1)(s+6)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+6}$$

$$Y(s) = \frac{60}{(s+1)(s+6)} = \frac{D}{s+1} + \frac{E}{s+6}$$

For X $\quad 60(s+3) = A(s+1)(s+6) + Bs(s+6) + Cs(s+1)$

set $s=0$ $\quad 60(3) = A(1)(6) \Rightarrow A = \frac{60(3)}{6} = 30$

$s=-1$ $\quad 60(2) = B(-1)(5) \Rightarrow B = \frac{60(2)}{-5} = -24$

$s=-6 \quad 60(-3) = C(-6)(-5) \Rightarrow C = \frac{60(-3)}{30} = -6$

The Y is done similarly. We get

$$X(s) = \frac{30}{s} - \frac{24}{s+1} - \frac{6}{s+6}$$

$$Y(s) = \frac{12}{s+1} - \frac{12}{s+6}$$

The solution to the system

$$x(t) = \mathcal{L}^{-1}\{X(s)\}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

$$\boxed{\begin{aligned} x(t) &= 30 - 24e^{-t} - 6e^{-6t} \\ y(t) &= 12e^{-t} - 12e^{-6t} \end{aligned}}$$

Example

Use the Laplace transform to solve the system of equations

$$\begin{aligned} x''(t) &= y, \quad x(0) = 1, \quad x'(0) = 0 \\ y'(t) &= x, \quad y(0) = 1 \end{aligned}$$

Let $X = \mathcal{L}\{x\}$ and $Y = \mathcal{L}\{y\}$

$\mathcal{L}\{x''\} = \mathcal{L}\{y\}$

$\mathcal{L}\{y'\} = \mathcal{L}\{x)$

$s^2 X - s x(0) - x'(0) = Y$

$sY - y(0) = X$

use $x(0) = 1$, $x'(0) = 0$

$y(0) = 1$

$$s^2 X - s = Y \qquad\qquad s^2 X - Y = s$$
$$sY - 1 = X \quad \Longrightarrow \quad -X + sY = 1$$

Using Crammer's rule

$$A = \begin{bmatrix} s^2 & -1 \\ -1 & s \end{bmatrix}, \quad A_x = \begin{bmatrix} s & -1 \\ 1 & s \end{bmatrix}, \quad A_y = \begin{bmatrix} s^2 & s \\ -1 & 1 \end{bmatrix}$$

$$\det(A) = s^3 - 1, \quad \det(A_x) = s^2 + 1, \quad \det(A_y) = s^2 + s$$

$$X(s) = \frac{s^2+1}{s^3-1} = \frac{s^2+1}{(s-1)(s^2+s+1)}$$

$$Y(s) = \frac{s^2+s}{s^3-1} = \frac{s(s+1)}{(s-1)(s^2+s+1)}$$

Note s^2+s+1 is irreducible

The partial fraction decomp looks like

$$X = \frac{s^2+1}{(s-1)(s^2+s+1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+s+1}$$

$$\varphi = \frac{s(s+1)}{(s-1)(s^2+s+1)} = \frac{D}{s-1} + \frac{Es+F}{s^2+s+1}$$

After some effort, we get

$$X = \frac{\frac{2}{3}}{s-1} + \frac{\frac{1}{3}(s-1)}{s^2+s+1}$$

$$\varphi = \frac{2/3}{s-1} + \frac{\frac{1}{3}(s+2)}{s^2+s+1}$$

We'd need to complete the square.

$$s^2+s+1 = s^2+s+\frac{1}{4}-\frac{1}{4}+1 = \left(s+\frac{1}{2}\right)^2+\frac{3}{4}$$

$$= \left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2$$

We need $s+\frac{1}{2}$ everywhere that s appears

$$s-1 = s+\frac{1}{2}-\frac{1}{2}-1 = s+\frac{1}{2}-\frac{3}{2}$$

$$s+2 = s+\frac{1}{2}-\frac{1}{2}+2 = s+\frac{1}{2}+\frac{3}{2}$$

$$X = \frac{2/3}{s-1}+\frac{1}{3}\,\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}-\frac{1}{3}\,\frac{3/2}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$Y = \frac{2/3}{s-1}+\frac{1}{3}\,\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}+\frac{1}{3}\,\frac{3/2}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

We'll Finish Next Time.

Crammer's Rule

Crammer's Rule is a method to solve an algebraic linear system of the form

$$\begin{array}{ccccc} ax & + & by & = & e \\ cx & + & dy & = & f \end{array}$$

Define the following matrices

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad A_x = \begin{bmatrix} e & b \\ f & d \end{bmatrix}, \quad \text{and} \quad A_y = \begin{bmatrix} a & e \\ c & f \end{bmatrix}.$$

If $\det(A) \neq 0$, then the system is uniquely solvable and the solution

$$x = \frac{\det(A_x)}{\det(A)} \quad \text{and} \quad y = \frac{\det(A_y)}{\det(A)}.$$

Crammer's Rule 3×3 case

$$\begin{array}{ccccccc} a_{11}x & + & a_{12}y & + & a_{13}z & = & b_1 \\ a_{21}x & + & a_{22}y & + & a_{23}z & = & b_2 \\ a_{31}x & + & a_{32}y & + & a_{33}z & = & b_3 \end{array}$$

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \qquad A_x = \begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix}$$

$$A_y = \begin{bmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{bmatrix}, \qquad A_z = \begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{bmatrix}$$

If $\det(A) \neq 0$, then the solution to the system

$$x = \frac{\det(A_x)}{\det(A)}, \quad y = \frac{\det(A_y)}{\det(A)}, \quad \text{and} \quad z = \frac{\det(A_z)}{\det(A)}.$$