November 8 Math 2306 sec. 53 Fall 2024

A Look Ahead: Solving IVPs

If $f(t)$ is defined on $[0, \infty)$, is differentiable, and has Laplace transform $F(s) = \mathscr{L}\{f(t)\}$, then

$$\mathscr{L}\{f'(t)\} = sF(s) - f(0)$$

Use this result to solve the initial value problem

$$y'(t) + 4y(t) = 16t, \quad y(0) = 1$$

we'll take $\mathscr{L}$ of the ODE. Let $\mathscr{L}\{y(t)\} = Y(s)$

$$\mathscr{L}\{y' + 4y\} = \mathscr{L}\{16t\}$$

$$y'(t) + 4y(t) = 16t, \quad y(0) = 1$$

$$\mathcal{L}\{y'\} + 4\,\mathcal{L}\{y\} = 16\,\mathcal{L}\{t\}$$

$$sY(s) - \underbrace{y(0)}_{=1} + 4Y(s) = 16\left(\frac{1}{s^2}\right)$$

Now, solve for $Y(s)$ using algebra.

$$sY(s) - 1 + 4Y(s) = \frac{16}{s^2}$$

$$sY(s) + 4Y(s) = \frac{16}{s^2} + 1$$

$$(s+4)\,Y(s) = \frac{16 + s^2}{s^2}$$

$$Y(s) = \frac{16+s^2}{s^2(s+4)}$$

This is the transform of the solution to the IVP

We want $y(t) = \mathcal{L}^{-1}\{Y(s)\}$.

We need a partial fraction decomp.

$$\frac{16+s^2}{s^2(s+4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+4}$$

Clear fractions

$$16+s^2 = As(s+4) + B(s+4) + Cs^2$$

$$s^2 + 0s + 16 = (A+C)s^2 + (4A+B)s + 4B$$

$$4B = 16 \Rightarrow B = 4$$

$$4A + B = 0 \Rightarrow A = \frac{-1}{4} B = -1$$

$$A + C = 1 \Rightarrow C = 1 - A = 2$$

$$Y(s) = \frac{-1}{s} + \frac{4}{s^2} + \frac{2}{s+4}$$

The solution $y(t) = \mathcal{L}^{-1}\{Y(s)\}$.

$$y(t) = \mathcal{L}^{-1}\left\{\frac{-1}{s} + \frac{4}{s^2} + \frac{2}{s+4}\right\}$$

$$= -\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + 4\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\}$$

$$y(t) = -1 + 4t + 2e^{-4t}$$

Section 15: Shift Theorems

Suppose we wish to evaluate $\mathscr{L}^{-1}\left\{\frac{2}{(s-4)^3}\right\}$. Does it help to know that $\mathscr{L}\left\{t^2\right\}=\frac{2}{s^3}$?

Note that by definition

$$e^{-st}\cdot e^{4t} = e^{-st+4t}$$

$$= e^{-(s-4)t}$$

$$\mathscr{L}\left\{e^{4t}t^2\right\} = \int_0^\infty e^{-st}e^{4t}t^2\,dt$$

$$= \int_0^\infty e^{-(s-4)t}\,t^2\,dt$$

Let $w = s-4$

$$= \int_0^\infty e^{-wt}\,t^2\,dt$$

$$= \frac{2!}{w^3} = \frac{2}{(s-4)^3}$$

Shift (or translation) in s.

Theorem: Suppose $\mathscr{L}\{f(t)\} = F(s)$. Then for any real number a

$$\mathscr{L}\left\{e^{at}f(t)\right\} = F(s-a).$$

We can state this in terms of the inverse transform. If $F(s)$ has an inverse Laplace transform, then

$$\mathscr{L}^{-1}\{F(s-a)\} = e^{at}\mathscr{L}^{-1}\{F(s)\}.$$

We call this a **translation** (or a **shift**) in s theorem.

Example:

Suppose $f(t)$ is a function whose Laplace transform[1]

$$F(s) = \mathscr{L}\{f(t)\} = \frac{1}{\sqrt{s^2+9}}$$

Evaluate

$$\mathscr{L}\left\{e^{-2t}f(t)\right\} = F(s-(-2)) = F(s+2)$$

$a = -2$

$$= \frac{1}{\sqrt{(s+2)^2+9}}$$

[1] It's not in our table, but this is an actual function known as a *Bessel function*.

Examples: Evaluate

(a) $\mathscr{L}\{t^6 e^{3t}\} = \dfrac{6!}{(s-3)^7}$.

- Ignore the e^{at}, find $\mathcal{L}\{f(t)\}$. $F(s) = \mathcal{L}\{t^6\} = \dfrac{6!}{s^7}$
- Identify a in e^{at}, $a = 3$

Find $F(s-a) = F(s-3)$

Examples: Evaluate

(b) $\mathscr{L}\{e^{-t}\cos(t)\}$ $= \dfrac{s+1}{(s+1)^2+1}$

$F(s) = \mathcal{L}\{\cos t\} = \dfrac{s}{s^2+1^2}$, $a=-1$, $F(s-(-1)) = F(s+1)$

(c) $\mathscr{L}\{e^{-t}\sin(t)\}$ $= \dfrac{1}{(s+1)^2+1}$

$F(s) = \mathcal{L}\{\sin t\} = \dfrac{1}{s^2+1^2}$, $a=-1$

$F(s+1)$

Inverse Laplace Transforms (completing the square)

(a) $\mathscr{L}^{-1}\left\{\dfrac{s}{s^2+2s+2}\right\}$

s^2+2s+2 is irreducible, we complete the square.

$$s^2+2s+1-1+2 = (s+1)^2+1$$

$$\frac{s}{(s+1)^2+1}$$ we need $s+1$ everywhere there's an s

Note $s = s+1-1$

$$\frac{s}{(s+1)^2+1} = \frac{s+1-1}{(s+1)^2+1} = \frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}$$

$$\mathscr{L}^{-1}\left\{\frac{s}{s^2+2s+2}\right\} = \mathscr{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}\right\}$$

$$= \mathscr{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1}\right\} - \mathscr{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}$$

looks like $\frac{s}{s^2+1}$ and $\frac{1}{s^2+1}$ with $s+1$ in place of s

$$= e^{-1t}\,\mathscr{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - e^{-1t}\,\mathscr{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= e^{-t}\cos t - e^{-t}\sin t$$