November 9 Math 2306 sec. 52 Fall 2022

Section 16: Laplace Transforms of Derivatives and IVPs

Transforms of Derivatives For $y = y(t)$ defined on $[0, \infty)$ having derivatives y', y'' and so forth, if

$$\mathscr{L}\left\{y(t)\right\} = Y(s),$$

then

$$\begin{aligned}
\mathscr{L}\left\{\frac{dy}{dt}\right\} &= sY(s) - y(0),\\
\mathscr{L}\left\{\frac{d^2y}{dt^2}\right\} &= s^2Y(s) - sy(0) - y'(0),\\
\mathscr{L}\left\{\frac{d^3y}{dt^3}\right\} &= s^3Y(s) - s^2y(0) - sy'(0) - y''(0),\\
&\vdots\\
\mathscr{L}\left\{\frac{d^ny}{dt^n}\right\} &= s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - y^{(n-1)}(0).
\end{aligned}$$

Solving IVPs

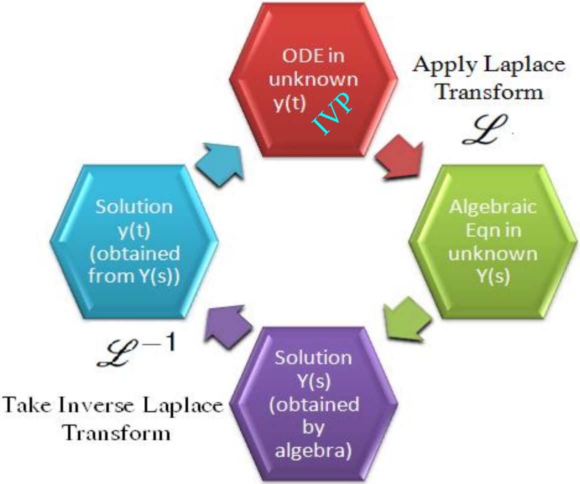


Figure: We use the Laplace transform to turn our DE into an algebraic equation. Solve this transformed equation, and then transform back.

Unit Impulse

Recall that the rectangular function $R_\epsilon(t)$ are zero for most t, but are defined so that the area under the curve is always a rectangle with area 1 so that

$$\int_{-\infty}^{\infty} R_\epsilon(t)\,dt = 1 \quad \text{for any } \epsilon > 0.$$

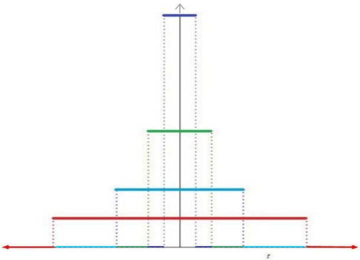


Figure: $R_\epsilon(t) = \begin{cases} \frac{1}{2\epsilon}, & |t| < \epsilon \\ 0, & |t| > \epsilon \end{cases}$

Unit Impulse

The Dirac delta *function*, denoted by $\delta(\cdot)$, models a strong instantaneous force. One way to define this function is as the limit

$$\delta(t) = \lim_{\epsilon \to 0} R_\epsilon(t).$$

This is not a function in the usual sense, but it has several properties.

- $\displaystyle\int_{-\infty}^{\infty} \delta(t-a)\,dt = 1$ for any real number a.
- $\displaystyle\int_{-\infty}^{\infty} \delta(t-a) f(t)\,dt = f(a)$ if a is in the domain of the function f.
- $\mathscr{L}\{\delta(t-a)\} = e^{-as}$ for any constant $a \geq 0$.

Remark: This is an example of what is called a *generalized function*, *generalized functional*, or *distribution*. In this context, it can be thought of as the derivative of the Heaviside step function. That is, for any $a \geq 0$

$$\frac{d}{dt}\mathscr{U}(t-a) = \delta(t-a).$$

Solve the IVP using the Laplace Transform

A 1 kg mass is suspended from a spring with spring constant 10 N/m. A damper induces damping of 6 N per m/sec of velocity. The object starts from rest from a position 10 cm above equilibrium. At time $t = 1$ second, a unit impulse force is applied to the object. Determine the object's position for $t > 0$.

The corresponding IVP for the situation described is

$$x'' + 6x' + 10x = \delta(t-1), \quad x(0) = 0.1, \quad x'(0) = 0$$

Let $X(s) = \mathcal{L}\{x(t)\}$.

$$\mathcal{L}\{x'' + 6x' + 10x\} = \mathcal{L}\{\delta(t-1)\}.$$

$$\mathcal{L}\{x''\} + 6\mathcal{L}\{x'\} + 10\mathcal{L}\{x\} = e^{-1s}$$

$$s^2X(s) - \underset{0.1}{sx(0)} - \underset{0}{x'(0)} + 6\left(sX(s) - \underset{0.1}{x(0)}\right) + 10X(s) = e^{-s}$$

$$s^2X(s) - 0.1s + 6sX(s) - 0.6 + 10X(s) = e^{-s}$$

$$(s^2+6s+10)X(s) - 0.1s - 0.6 = e^{-s}$$

$$(s^2+6s+10)X(s) = e^{-s} + 0.1s + 0.6$$

matches characteristic poly

$$X(s) = \frac{e^{-s}}{s^2+6s+10} + \frac{0.1s+0.6}{s^2+6s+10}$$

$s^2+6s+10$ is irreducible $\Rightarrow$ Complete the square

$$s^2+6s+10 = s^2+6s+9-9+10 = (s+3)^2+1$$

$$X(s) = \frac{e^{-s}}{(s+3)^2+1} + \frac{0.1s+0.6}{(s+3)^2+1}$$

$$0.1s+0.6 = 0.1(s+3-3)+0.6 = 0.1(s+3)+0.3$$

$$X(s) = \frac{e^{-s}}{(s+3)^2+1} + 0.1\frac{s+3}{(s+3)^2+1} + 0.3\frac{1}{(s+3)^2+1}$$

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathcal{U}(t-a) \quad \text{for } f(t) = \mathcal{L}^{-1}\{F(s)\}$$

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at}\mathcal{L}^{-1}\{F(s)\}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2+1}\right\} = e^{-3t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = e^{-3t}\sin t = f(t)$$

$$\mathcal{L}^{-1}\left\{\frac{s+3}{(s+3)^2+1}\right\} = e^{-3t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = e^{-3t}\cos t$$

$$\mathcal{L}^{-1}\left\{e^{-s}\frac{1}{(s+3)^2+1}\right\} = f(t-1)\,\mathcal{U}(t-1)$$

$$X(s) = \frac{e^{-s}}{(s+3)^2+1} + 0.1\,\frac{s+3}{(s+3)^2+1} + 0.3\,\frac{1}{(s+3)^2+1}$$

The solution to the IVP $x(t) = \mathcal{L}^{-1}\{X(s)\}$

$$x(t) = e^{-3(t-1)}\sin(t-1)\,\mathcal{U}(t-1) + 0.1\,e^{-3t}\cos t + 0.3\,e^{-3t}\sin t$$

Transfer Function and Weight Function

For the constant coefficient, linear ODE

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = g(t)$$

with characteristic polynomial

$$P(s) = a_n s^2 + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$$

the function

$$\boxed{W(s) = \frac{1}{P(s)} \quad \text{is called the } \textbf{transfer function.}}$$

Transfer Function and Weight Function

The transfer function is the Laplace transform of what is known as the weight function for the ODE.

$$w(t) = \mathscr{L}^{-1}\left\{\frac{1}{P(s)}\right\} \quad \text{is called the } \textbf{weight function.}$$

The weight function is the solution to the IVP that has the Dirac delta on the right side and all initial conditions are zero. That is, $w(t)$ solves

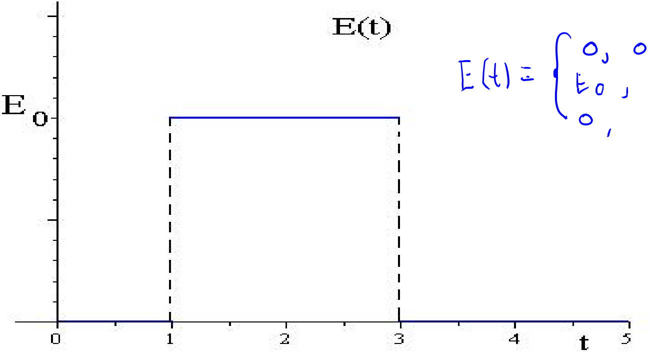
$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = \delta(t),$$

subject to initial conditions

$$y(0) = 0, \quad y'(0) = 0, \quad \dots, \quad y^{(n-1)}(0) = 0$$

Solve the IVP

An LR-series circuit has inductance $L = 1$h, resistance $R = 10\Omega$, and applied force $E(t)$ whose graph is given below. If the initial current $i(0) = 0$, find the current $i(t)$ in the circuit.



$$E(t) = \begin{cases} 0, & 0 \le t < 1 \\ E_0, & 1 \le t < 3 \\ 0, & t \geq 3 \end{cases}$$

$$E(t) = 0 - 0\,\mathcal{U}(t-1) + E_0\,\mathcal{U}(t-1) - E_0\,\mathcal{U}(t-3) + 0\,\mathcal{U}(t-3)$$

LR Circuit Example

$$L\frac{di}{dt} + Ri = E$$

The IVP is

$$\frac{di}{dt} + 10i = E_0 \mathcal{U}(t-1) - E_0 \mathcal{U}(t-3), \quad i(0) = 0$$

We'll solve this next time.